J. Zhao, A Smith inequality for fixed point Floer cohomology

(I) Classical Smith ineq. for $\mathbb{Z}/p\mathbb{Z}$ action.

Fix p prime, $G = \mathbb{Z}/p\mathbb{Z} \curvearrowright X$ finite CW cplx.

X^G = fixed point set.

Smith inequality: If $\mathbb{K} = \mathbb{F}_p$, then $\sum_i \dim H^i(X;\mathbb{F}_p) \geq \sum_i \dim H^i(X^G, \mathbb{K})$.

2 consequences:

- If X is $\mathbb{F}_p$ acyclic ($H^*(X, \mathbb{K}) = H^*(pt, \mathbb{K})$) then so is X^G.
- " " $\mathbb{F}_p$ - homology sphere, then " so is X^G.

inequality + a spectral sequence

$\Rightarrow X^G \neq \emptyset$.

(II) Fixed-point Floer cohomology $HF^*(\phi)$

Setup: Given a Liouville domain (M, θ), ϕ exact symplectomorphism. Assume: $\phi|_{\partial M} = id$,
& that the fixed points of ϕ, ϕ^p are non-degenerate. (& all ϕ^{p^k} too.)

Then, $HF^*(\phi)$ is the Floer homology of

$\mathbb{Z}/2$-grading

$$\mathcal{A}_\phi : \mathcal{L}_\phi M \longrightarrow \mathbb{R}$$

$$\mathcal{L}_\phi M = \{\gamma : \mathbb{R} \to M \mid \gamma(t) = \phi(\gamma(t+1))\}$$

(need to small positively perturb to eliminate fixed points at ∞).

crit $\mathcal{A}_\phi \overset{1:1}{\longleftrightarrow} \mathrm{Fix}(\phi)$

$CF^*(\phi) = \mathbb{K}\langle \text{fixed pts. of } \phi\rangle$

Differential counts hol. sections of

$$\begin{array}{l} \mathbb{R}\times M_\phi \cong \mathbb{C}\times M /_{(s,t,x)\to(s,t+1,\phi(x))} \\ \downarrow \\ \mathbb{R}\times S^1 \end{array}$$

For fixed $k \in \mathbb{N}$, there is a $\mathbb{Z}/k\mathbb{Z}$ action on $\mathcal{L}_{\phi^k} M = \{\gamma : \mathbb{R} \to M \mid \gamma(t) = \phi^k(\gamma(t+k))\}$

gen.

$\langle \sigma\rangle \in \mathbb{Z}/k\mathbb{Z}$, $(\sigma \cdot \gamma)(t) = \phi(\gamma(t+1))$. (note: $\sigma^k \cdot \gamma(t) = \phi^k(\gamma(t+k)) \overset{?}{=} \gamma(t)$?).

Thm [in progress]: for any fixed prime p, $\mathbb{K}=\mathbb{F}_p$, we have

$$\sum_i \dim_{\mathbb{K}} HF^i(\phi^p) \geq \sum_i \dim_{\mathbb{K}} HF^i(\phi) \, .$$ (but not the degree-wise.),

Rmks: ① In the case $p=2$, proved by Seidel, Hendricks. ← requires assumptions / uses Seidel-Smith localization.

② Inequality fails for k non-prime
(at the moment, not clear why).

Corollary: Given M, $\phi \in Symp^{ex}(M)$; if $\dim_k HF^*(\phi) > \dim_k H^*(M)$ $(*)$,

then $[\phi], [\phi^p], \ldots, [\phi^{p^k}], \ldots$, are non-trivial in $Symp^{ex}(M)$.

(might hope any order of $[\phi]^s$ is non-trivial if $(*)$ holds for any $\mathbb{K}=\mathbb{F}_q$).

Rmk: $HF^*(\phi) = HF^*(\Delta, G_\phi)$ $\Delta \subseteq M^- \times M$.

(III) An outline of the proof

2 ingredients:

① Define $\mathbb{Z}/k\mathbb{Z}$-equivariant Floer cohomology $HF^*_{\mathbb{Z}/k\mathbb{Z}}(\phi^k)$

② For p prime, define a "p^{th} power" map": (which should be an version of Tate homologies) ← Chas-Sullivan

heuristically

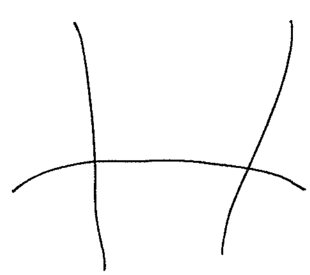
$$\mathcal{L}M \xrightarrow{\Delta} \underbrace{\mathcal{L}M \times \cdots \times \mathcal{L}M}_{p} \xrightarrow{*CS} \mathcal{L}M$$

$$a \longmapsto \underbrace{a * \cdots * a}_{p \text{ times}}$$

For ①, Given M, $\phi \in Symp^{ex}(M)$,

M_ϕ admits a stable Hamiltonian structure $(\tilde{\omega}, \lambda = dt)$; Reeb v.f. $R=\partial_t$,

& $Fix(\phi^k) \longleftrightarrow$ Reeb orbits of period k in M_ϕ.



$\mathbb{R}\times M_\phi$

To define $CF^*(\phi^k)$, one can choose a $\mathbb{Z}/k\mathbb{Z}$-symmetric data, $J_t \in \mathcal{J}_{\phi^k} = \{J_t = (\phi^k)_* J_{t+k}\}$ periodic J_t's;

note ~~satisfying~~, for $\sigma \in \mathbb{Z}/k\mathbb{Z}$, $(\sigma \cdot J_t)(t) = \phi_* J_{t+1}$).

Symmetric J_t are invariant under this action of $\mathbb{Z}/k\mathbb{Z}$.

If can make such a choice, then

$\exists$ a $\mathbb{Z}/k\mathbb{Z}$-action on $CF^*(\phi^k)$, & it is a <u>strict</u> action.

(e.g., for $k=2$, $\sigma \circlearrowright CF^*(\phi^2)$, $\sigma^2 = id$ holds for $\mathbb{Z}/2\mathbb{Z}$-sym-a.c.s.)

In this case, define

$$CF^*_{eq}(\phi^k) = H^*(\mathbb{Z}/k\mathbb{Z}; CF^\bullet(\phi^k)) = \{CF^*(\phi) \xrightarrow{id-\sigma} CF^*(\phi) \xrightarrow{id+\sigma} \cdots\}$$

$\uparrow$ group coh.

In general, $\sigma^2 - id = dh + hd$, & h & higher homotopy terms give more differentials on).

For general $k \in \mathbb{Z}$,

$$H^*(\mathbb{Z}/k\mathbb{Z}; CF^\bullet(\phi^k)) = \{CF^\bullet(\phi^k) \xrightarrow{id-\sigma} CF^\bullet(\phi^k) \xrightarrow{id+\sigma+\cdots+\sigma^{k-1}} CF^\bullet(\phi^k) \to \cdots\}$$

Filtering by columns, this gives a spectral sequence converging to $HF^*_{eq}(\phi^k)$ w/

$p=2$ $\quad \ker(d : E_1^{pq}) = HF^\bullet(\phi^2)^{\mathbb{Z}/2}$ $\quad$ invariant part. $\quad$ (so rk of $HF^\bullet(\phi^2)^{\mathbb{Z}/2} \geq$ rk $HF^\bullet_{\mathbb{Z}_2}(\phi^2)$,

(general p: odd & even is different)

$$\Rightarrow \sum_i \dim_{\mathbb{K}} HF^*(\phi^2)^{\mathbb{Z}/2} \geq \sum_i \dim_{\mathbb{K}[\hbar]} HF^*_{eq}(\phi^2)$$

$= H^\bullet(\mathbb{RP}^\infty)$

So, if one had a <u>localization theory</u>

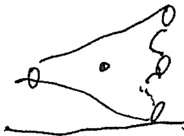
then we'd be done: invert $\hbar$, & this would become $HF^\bullet(\phi)((\hbar))$.

Problem: <u>don't have localization generally</u>, in this setting.

So, instead, want to relate $\dim_{\mathbb{K}[\hbar]} HF^\bullet_{eq}(\phi^2)$ to $\dim_{\mathbb{K}} HF^\bullet(\phi)$, at least after inverting $\hbar$.

p prime now.

(2) Consider



$$\begin{array}{ccccc} P_{eq}: CF^*(\phi) & \xrightarrow{\text{Tate map}} & CF^*(\phi)^{\otimes p} & \xrightarrow{\tilde{P}_{eq}} & CF^*(\phi^p) \\ \circlearrowleft & & J_{\mathbb{Z}/p\mathbb{Z}} & & \circlearrowleft \mathbb{Z}/p\mathbb{Z}. \\ (\mathbb{Z}/p) & x \longmapsto & \underbrace{x\otimes \cdots \otimes x}_{p \text{ times}} & & \\ (\text{trivial}) & & & & \end{array}$$

Prop: The composition induces a quasi-isomorphism on $\mathbb{Z}/p$-equivariant cohomology after inverting the equivariant parameters, (tensoring w/ $\check{H}^*(B\mathbb{Z}_p;\mathbb{F}_p)$ Tate cohomology)

This $\Rightarrow$ $\mathrm{rk}_{H^*_{Tate}(\mathbb{Z}/p;\mathbb{K})} HF^*_{eq}(\phi^p) \geq \mathrm{rk}\, HF^*(\phi)$, as desired.

Note/Rmk. that the Tate map is bad on the chain level (not additive), but it's ok/additive on cohomology & induces an isomorphism on Tate cohomology groups.

So, it remains to define the p^{th} power map.

- Constructing $\tilde{P}_{eq}$: Consider the p-fold branched cover

$$\begin{array}{ccc} \Sigma & \longrightarrow & \mathbb{C}^* \setminus \{\text{roots of unity}\} \circlearrowleft \mathbb{Z}/p\mathbb{Z}, \\ \downarrow & & \downarrow z^p = 1-y \\ \mathbb{R}\times S^1 & \xrightarrow{\exp(s+it)} & \mathbb{C}^* \end{array}$$

branched at 1 point; with no moduli.

$\tilde{P}_{eq}$ counts $u: \Sigma \longrightarrow \mathbb{R}\times M_\phi$;

$\|$

$P_0 + hP_1 + \cdots$ $\quad$ P_1 & P_2 etc. measure failure of P_0 to be $\mathbb{Z}/p\mathbb{Z}$ equivariant on chain level. (failure of equivariant transversality)

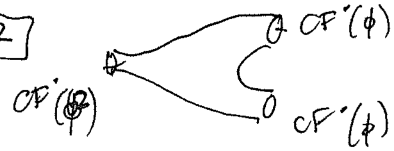
still have to deal w/ constant branched cylinders, where can't do domain dependent J. Non-transverse.

To deal with this, can do some obstruction bundle gluing.

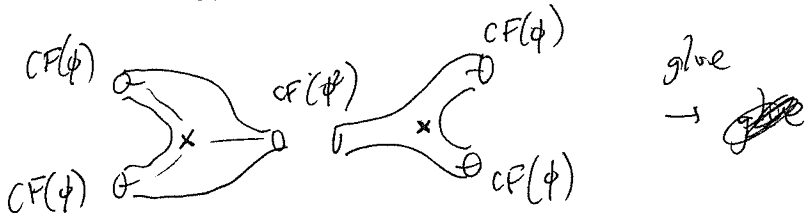
To see that

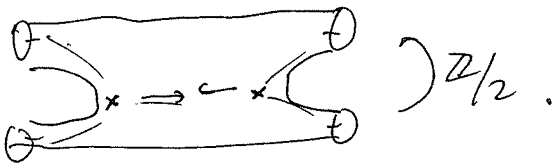
$\tilde{P}_{eq}$ is a quasi-isomorphism on (Tate) homology:

$\boxed{p=2}$



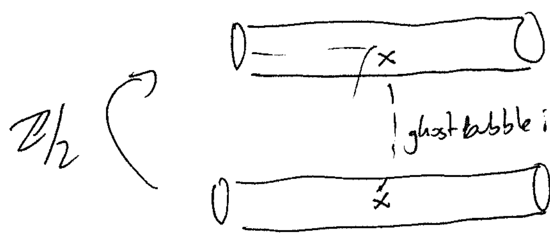
, construct a "coproduct"





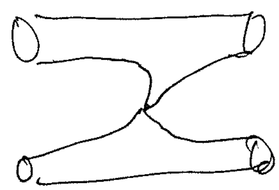
degen.

bubble comes out, but constant (exact setting).



ghost bubble identifies two pts. or

(monotone case: sphere arises in codim. 1).



What is this? This is a $\mathbb{Z}/2$-equivariant to

~~ghost bubble~~ cap product with P.D. to $[\Delta] \in H^*_{\mathbb{Z}/2}(M \times M)$.

But we classically know that on Tate cohomology, $[\Delta]$ is invertible

($\mathbb{Z}/2$-equiv. Euler class when restricted to fixed locus). $\boxtimes$