9/3/2015 M. Abouzaid

- math.columbia.edu/mcduff70 (Mar. 2016 $\exists$ funding for jr. participants)
- schms.math.berkeley.edu — Jobs (postdocs @ Columbia, ...) — Nov. 5–8 (Conference at UPenn)

Floer theory & loop homology.

Q closed, smooth mfld.

T^*Q exact symplectic (Liouville) $(\lambda = p\,dq \rightsquigarrow X_\lambda = p\,\partial_p)$.

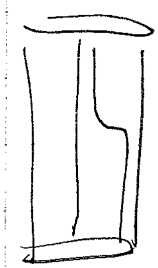
OR: (X any Liouville, w/ $Q \hookrightarrow X$ exact Lagn emb.)

$\mathcal{W}(T^*Q)$ wrapped Fukaya cat.

objects: $L \subset T^*Q$ exact & conical at ∞ (i.e. inv't under X_λ outside cpct.)

get a "Leg ∂."

Main property of $\mathcal{W}(T^*Q)$ is that the quasi-equiv. class of L is invariant under all Ham. diffeomorphisms which are conical at ∞ (& more is true too).

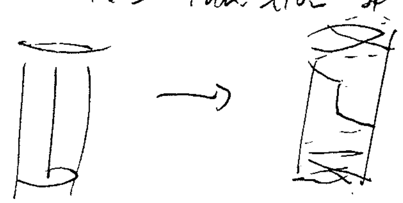
e.g.) 

To achieve this, we define

$$\mathrm{Hom}_{\mathcal{W}}(L_0, L_1) \equiv CF^*(L_0, L_1; H) \quad \text{agrees w/ } |p|^2 \text{ outside cpct. set.}$$

(but there are other ways of doing this!).

(i.e. the flow of Ham. flow of H



Goal: Compute $\mathcal{W}(T^*Q)$; more precisely, build a category

$\mathrm{Tw}\,\mathcal{W}(T^*Q)$ of "twisted complexes"; formal complexes

built from objects & morphisms of $\mathcal{W}(T^*Q)$.

e.g. $L_0 \xrightarrow{x} L_1$ $\quad x \in \mathrm{Hom}_{\mathcal{W}}(L_0, L_1)$ closed, $(dx=0)$ is in ob $Tw\,\mathcal{W}(T^*Q)$.

or:

$$\begin{tikzcd} L_0 \arrow[r, "x_{01}"'] \arrow[rr, bend left, "x_{02}"] & L_1 \arrow[r, "x_{12}"'] & L_2 \end{tikzcd}$$

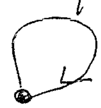
$dx_{01} = 0$

$dx_{12} = 0$,

& $dx_{02} = x_{01} \cdot x_{12}$ (keeps track of "expository/ product $= 0$ up to a homotopy").

An ~~…~~ enlargement of this called the "idempotent closure"
$Tw^{\pi}\mathcal{W}(T^*Q)$
where we ~~…~~ also add formal summands of idempotents $i \in \mathrm{Hom}_{Tw\,\mathcal{W}}(T, T)$
(s.t. $i^2 = i$, $di = 0$).
(e.g. like adding projective modules, not just free modules over a ring).
& we want to compute $Tw^{\pi}\mathcal{W}(T^*Q)$.
Philosophy: it may be difficult to identify a priori which objects of $Tw^{(\pi)}\mathcal{W}(T^*Q)$
correspond to embedded Lagrangians.

(I) Consider $\Omega_q Q$ (q is basepoint).



by concatenation, we have a map $\Omega_q Q \times \Omega_q Q \to \Omega_q Q$
So passing to chains, we have a dga
want $\longrightarrow$ $C_{-*}(\Omega_q Q)$ ($\mathbb{Z}$ coeffs if not specified).
$|d| = +1$.

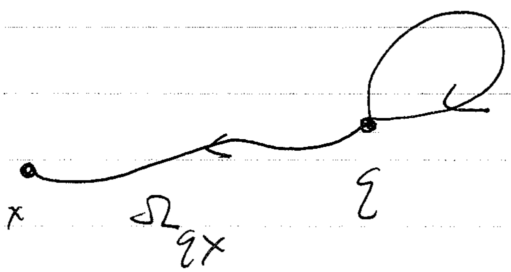
Thm: (A '10): $\exists$ an A_∞ functor
$$\mathcal{W}(T^*Q) \xrightarrow{\mathcal{V}} \mathrm{mod}\, C_{-*}(\Omega_q Q) \quad \text{(fully faithful.)}$$
factors through $Tw\, C_{-*}(\Omega_q Q)$ $\leftarrow$ factoring omitted from now

To construct a module $\mathcal{V}L$ associated to $L \in Ob(\mathcal{W})$,

consider

shift by Maslov index of intersection points.

$$\mathcal{V}L = \left(\bigoplus_{x \in Q \cap L} C_{-*}(\Omega_{qx} Q)[|x|], \; d \right).$$



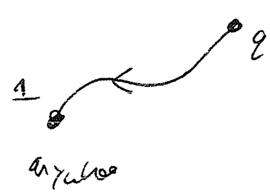
$\exists$ map $\Omega_q Q \times \Omega_{qx} Q \xrightarrow{\text{concat.}} \Omega_{qx}$

<u>in fact:</u> $C_{-*}(\Omega_{qx} Q) \underset{q.i.}{\cong} C_{-*}(\Omega_q Q)$

as modules over $C_{-*}(\Omega_q Q)$,

b/c have fibration

$$\begin{array}{ccc} \Omega_{qx} Q & \longrightarrow & P_q Q \\ \downarrow & & \downarrow ev_1 \\ x & \longrightarrow & Q \end{array}$$



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d has a classical component: which is the differential on $C_{-*}(\Omega_{qx} Q)$.

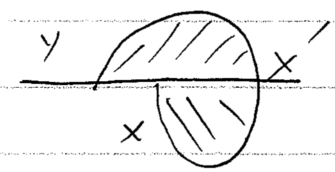
Let $x, y \in L \pitchfork Q$. (assume $\pitchfork$, otherwise perturb Hamiltonians to achieve it)

Gromov-Floer compactification $\rightarrow$ $\overline{\mathcal{M}}(y, x) = \left\{ \right.$  $\left. \right\}$ (mod $\mathbb{R}$).

$\downarrow ev$ (evaluate along ∂ contained in Q) (essentially unique way to do this, as $Diff([0,1])$ contractible; make a choice)

$\Omega_{xy} Q$.

Topological mfld w/ ∂, s.t. $\partial \overline{\mathcal{M}}(x, y) = \bigcup \overline{\mathcal{M}}(y, x') \times \overline{\mathcal{M}}(x', x)$.

(reflecting breaking: )

Pick fundamental chains

$$[\mathcal{M}(y,x)] \in C_{|x|-|y|+1}(\overline{\mathcal{M}}(x,y),\mathbb{Z})$$

such that $\partial [\mathcal{M}(y,x)] = \sum_{x'} [\mathcal{M}(y,x')] \times [\mathcal{M}(x',x)]$

"exterior product on chains."

By evaluation,

$$ev_*[\mathcal{M}(y,x)] \in C_{|x|-|y|+1}(\Omega_{xy}Q).$$

Multiplying with $ev_*[\mathcal{M}(y,x)]$ gives a map $C_{-*}(\Omega_{qx}Q) \to C_{-*}(\Omega_{qy}Q)$.

of degree $= \dim \mathcal{M}(y,x)$, becomes degree 1

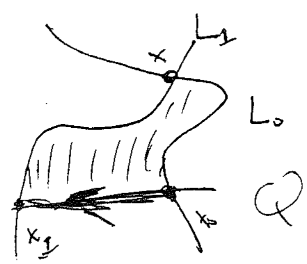
~~which~~ after shifting.

This is the component of diff'l going from x summand of VL to y summand of VL.

<u>Morphisms</u>? E.g.,

On Floer homology groups, we have a map

$$\mathrm{Hom}_{\mathcal{W}}(L_0,L_1) = CF^*(L_0,L_1) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(C_{-*}\Omega_{qx_0}, C_{-*}\Omega_{q_1})$$



For each $x_0 \in L_0 \cap Q$, $x_1 \in L_1 \cap Q$..

Consider $\mathcal{M}(x, x, x_0) \xrightarrow{ev} \Omega_{x_0x_1}Q$

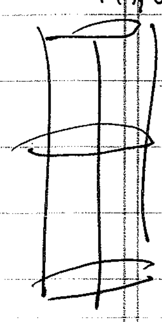
& repeat.

~~and~~ Induces a map $VL_0 \to VL_1$ of modules over $C_{-*}\Omega_q Q$.

& so on.

for $L = T_q^* Q$, we ~~can~~ obtain a map

$CF(T_q^*Q, T_q^*Q, H)$



$$\mathrm{Hom}_{\mathcal{W}}(T_q^*Q, T_q^*Q) \longrightarrow C_{-*} \Omega_q Q.$$

__Thm 2__: This is a quasi-isomorphism (if Q is __Spin__.).

Time-1 chords of Ham w/ both endpts. on T_q^*Q $\xleftrightarrow{1:1}$ {geodesics in Q, based at $\{q\}$}

$\downarrow$

compute homology of $\Omega_q Q$.

__Thm 3__: $\mathcal{W}(T^*Q)$ is ~~generated~~ split gen. by a cotangent fibre T_q^*Q.

(doesn't require any Maslov index assumption; or exactness even)

__meaning__:

$$\begin{array}{ccc} \mathcal{W}(T^*Q) & \longrightarrow & Tw^{\pi} \mathcal{W}(T^*Q) \\ \nearrow & & \uparrow \text{ is } \underline{\text{essentially}}\ \underline{\text{surjective}} \\ \mathcal{E} = \mathrm{Hom}_{\mathcal{W}}(T_q^*Q, T_q^*Q) & \longrightarrow & Tw^{\pi} \mathcal{E} \end{array}$$

$0 \to L \to 0$

$\wr\wr$

(summand of complex built from T_q^*Q).

Since $\mathcal{E} \simeq C_{-*} \Omega_q Q$, we conclude that $Tw^{\pi} \mathcal{W}(T^*Q) \simeq Tw^{\pi} C_{-*}(\Omega_q Q)$.

(aside: algebra on RHS $\Rightarrow Tw^{\pi} \simeq Tw$).

__Consequences__: • $Q \supset M$ submfld

contained in a contractible set (for simplicity; up to htpy need $M \subset Q$ contractible)

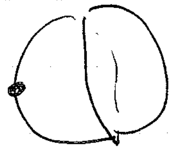
• $T_M^* Q \subset T^*Q$ conormal bdle. object of $\mathcal{W}(T^*Q)$

__Cor__: Under assumption, this is equivalent to "$C_* M \otimes T_q^* Q$"

in practice, if M ~~…~~ is a sphere$_q$ of dim. k,
then

$$T_M^* Q \simeq T_q^* Q \oplus T_q^* Q[k]$$

e.g., T^*S^2



(T^*S^2; Polterovich construction non-exact L disjoint from Q; is gen. by T_q^*Q but somehow maps non-trivially under $\mathcal{V}$?!)

(Rmk: $\widehat{\mathcal{W}}_{(T^*Q)}$ non-exact $\not\to$ mod $C_{-*}\Omega_q Q$.

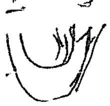
$\searrow$ mod $\widehat{C}_*(\Omega_q Q)$ ← some completion w.r.t. length / index filtration

↑ this completion explains why Polterovich [can] be explained in this setting)

(eventually: maybe want to ~~build~~ build categories w/ action filtrations homology algebra built).

$\mathrm{End}(T_q^*Q) \xrightarrow{\text{Yoneda}} \mathrm{Ext}_{C^*(Q)}(k, k)$ $\quad QH^{2n}$ has a 2n class, & $\otimes$ class-trivial.

$SH^* \otimes HH_*(\mathcal{E}_{pt}) \to \mathbb{K}$

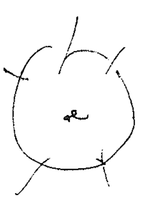


1 ∃?

$\hom(T_q^*Q, Q) \longrightarrow \mathrm{Ext}_{C^*(Q)}(k, C^*(Q)) \simeq \mathbb{K}^\vee.$ $\quad \langle e, \sigma \rangle = 1,$

σ deg $-n$.. $\quad HH_*(A, A) \otimes HH_*(B, B) \longrightarrow \mathbb{K}$ be degree-wise non-degenerate $\quad$ say non-deg.

in particular $\quad QH^{2n} \otimes HH_*(B,B) \to \mathbb{K}$ $\quad SH^* \otimes HH_*(B,B) \to \mathbb{K}$



is the CY str.

$\langle e_A, - \rangle : HH_*(B, B) \longrightarrow \mathbb{K}$ $\quad QH^0 \otimes HH_*(B,B) \to \mathbb{K}$

$\mathcal{Y}^r_K \otimes_e \mathcal{Y}^l_K$

$\langle e_A, - \rangle : HH_*(B, B) \longrightarrow \mathbb{K}$ $\quad \downarrow \times$ hits 1.

is equal to $\quad \hom(e, e)$

$V \to W$ $\quad O\mathcal{E}$

hits 1 = surjective. $\quad$ but: $\hom(e,e) \otimes (\mathcal{Y}^r_K \otimes_e \mathcal{Y}^l_K) \xrightarrow{\deg n} \mathbb{K}$ perfect. ∃ σ w/ 1

$V \otimes W \to R.$ "hits 1" $\quad \hom(e,e)^\vee[n] \simeq \mathcal{Y}^r_K \otimes_e \mathcal{Y}^l_K = \mathrm{End}_{\mathcal{B}/e}(K)$

$V \to (W \to R)$