Borman  Overtwisted contact structures  w/ Eliashberg–Murphy

Almost contact structure:

$$(M^{2n+1}, \xi = \ker \alpha)$$

$\omega$ non-deg. 2 form on $\xi$.  $\alpha \wedge \omega^n \neq 0$.

Overtwisted contact structure: Niederkrüger–Presas '09 (Casals–Murphy–Presas):

$$\left( \mathbb{D}^2_{OT} \times \mathbb{R}^{2n-2}, \ker(\alpha_{OT} + u\,d\phi) \right) \hookrightarrow (M^{2n+1}, \xi).$$

$\mathbb{D}^2_{OT} = \{z = 0\}$, $\alpha_{OT} = \cos(u)\,dz + \sin(u)\,d\phi$.

codim. 1 embedding, or thicken to $\mathbb{R}^3_{OT}$,

Thm: (B–E–M.)

$$\mathrm{Cont\,Str}_{OT}(M) \longleftrightarrow \mathrm{Almost\,Cont\,Str}(M)$$

is an isomorphism on $\pi_0$..

(restricting more data, it's a weak homotopy equiv.).

surjective: any a.cont. is rep. by a genuine str.

injective: Overtwisted ctct strs. are isomorphic if in same htpy class.

Follow-up work:

Casals–Murphy–Presas: • gave usable defn's of OT. (h-principle.)

• Generalized many of criteria for OT in 3-D. to higher dimensions.

• Connection w/ Plastikstufe.

Eliashberg–Murphy: Proved an h-principle for symp. structures on cobordisms $W^{2n}$ w/ an OT neg. end, when $2n > 4$.

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Gromov's h-principle:

If $V^{2n+1}$ is open, then $\mathrm{Cont\,Str}(V) \xrightarrow[\text{h.e}]{\simeq} \mathrm{Almost\,Cont}(V)$

(b/c V can be retracted onto a codim. 1 skeleton; "holonomic approx." gives in a small nbhd of codim 1.)

For closed manifolds:

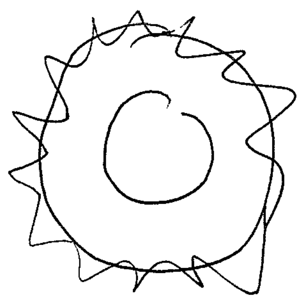
$(M^{2n+1}, \xi)$ almost contact structure.

Applying Gromov to a nhood of codim. 1 skeleton, we have that $\xi$ is homotopic to another $\xi'$ s.t. $(\xi')$ is genuine outside a finite # of balls $B_1, \dots, B_N \subset M$.

(Rmk: can take $N=1$, but then need to refine to a std. problem by choosing a cube of this ball anyway)

Left w/ following problem:

$(B^{2n+1}, \xi_{almost})$ and $\xi$ is genuine on a nhood of $\partial B$.



How can we make $\xi$ genuine while fixing $\xi$ near $\partial B$?

Call this a contact shell.

Overview of pf:

(1) Find a special contact str. $(B_{OT}, \xi_{OT})$ on a ball s.t. where you have an explicit way to show

$(B_{OT} \# B, \xi_{OT} \# \xi)$ equiv. to a genuine contact str.

where $(B, \xi)$ is a shell.

maybe of a certain fixed type.

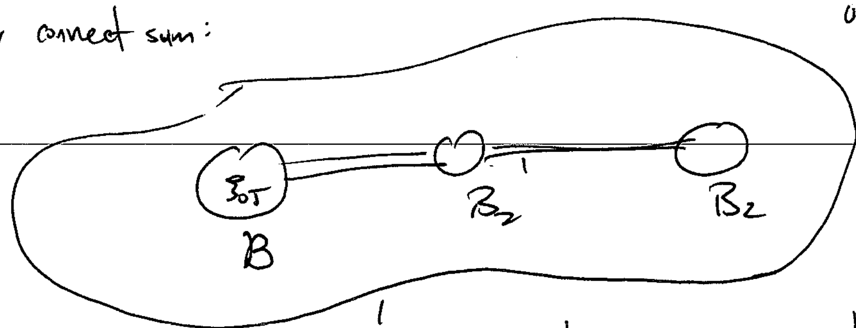
② Apply Gromov's theorem in a controlled way to show that

$(M^{2n+1}, \xi) \simeq (M^{2n+1}, \xi')$ where $\xi'$ is a genuine outside that
Almost contact

\# of balls, $(B_{OT}, \xi_{OT})$, and $\underbrace{(B_1, \xi'), \ldots, (B_N, \xi')}_{\text{contact shells of the "certain fixed type"}}$.

special type $\subseteq (M, \xi')$.

Apply connect sum:




Homotope $\xi_{OT} \# \xi' \# \cdots$ to a genuine structure rel. $\partial(B \# B_1 \# \cdots)$.

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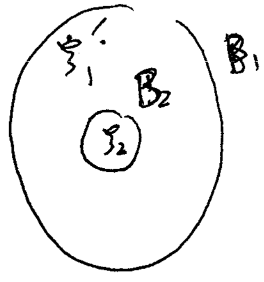
Domination of contact shells:

$(B_1, \xi_1)$ dominates $(B_2, \xi_2)$ if

$B_2 \hookrightarrow B_1$




where $\xi_1'$ agrees w/ $\xi_1$ near $\partial B_1$
& is genuine on $B_1 \setminus B_2$.

Hamiltonian contact shells:

ball $\Delta^{2n-1} \subseteq (\mathbb{R}^{2n-1}, \xi_{std})$.

$K : \Delta \longrightarrow \mathbb{R}$ w/ $K|_{\partial\Delta} > 0$.

Form a contact shell $(B_K, \eta_K)$. Near boundary, $\eta_K$ looks like:

$\partial B_K = \Sigma_K^1 \cup \Sigma_K^2$

& $\Sigma_K^1 = \{ (x, v, t) \in \Delta \times T^*_{v,t} S^1 : v = K(x) \}$.

$\eta_K$ looks like $\ker(\partial_{st} + v\,dt)$

$$\Sigma_k^2 = \{(x,v,t) \in \partial\Delta \times \mathbb{R}^2 : 0 \le v \le k(x)\}$$

$v = x^2+y^2$
$t$ = angle

? no solution if $k$ pos.-everywhere?

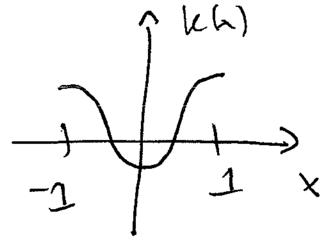
$\eta_k$ is just $\partial_{st} + v\,dt$.

Characteristic distribution is:

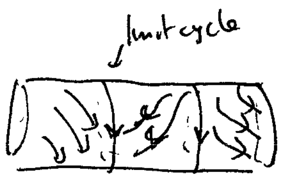
$-\partial_t + X_k$ on $\Sigma_k^1$, & $v\partial_v$ on $\Sigma_k^2$.

↑ contact v.f. assoc. to $k$

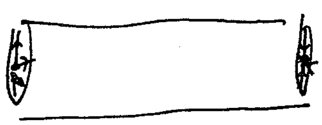
Example: $\Delta = [-1, 1] \subseteq \mathbb{R}^1$.




$\Sigma_k^1$



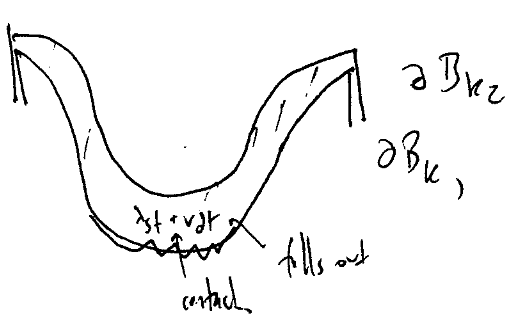

$\Sigma_k^2$



We see a piecewise smooth OT 2-disk:

Nice aspects:

1) Play well with domination. $k_1, k_2 : \Delta \to \mathbb{R}$ and $k_1 \le k_2$, then $B_{k_1}$ is dominated by $B_{k_2}$.



2) Applying contactomorphisms of $\Delta$ can affect the order of $K$ in non-trivial ways.

$$\Phi \in \mathrm{Cont}(\Delta) \qquad \Phi^{*}\alpha = c_{\Phi}\,\alpha .$$

Then $\Phi_{*} K = (c_{\Phi} \cdot K)\circ\Phi^{-1}$.

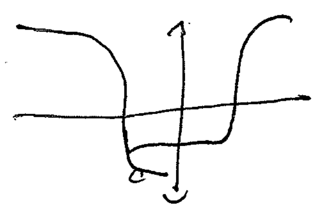
Then $\tilde{\Phi} : (B_K, \eta_K) \xrightarrow{\cong} (B_{\Phi_* K}, \eta_{\Phi_* K})$

gives freedom beyond looking at fixed Hamiltonian;
can use to get domination

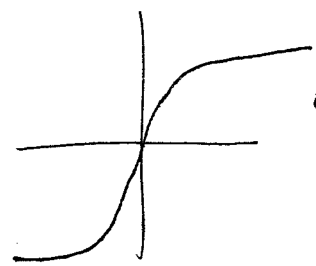
Applying this in dim. 3:

Lemma: ~~Id~~ Let $\Delta = [-1,1]$.
If $K : \Delta \to \mathbb{R}$ is neg. somewhere, then ~~$B_K$~~ $B_K$ is dominated by any other $B_{K'}$ for $K' : \Delta \to \mathbb{R}$.

Pf: Say $K$:

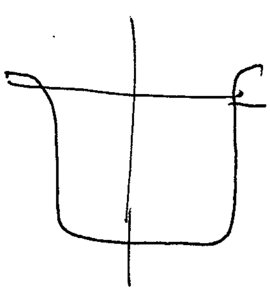


& let $\Phi$ be contacto:



$\in \mathrm{Cont}(\Delta)$

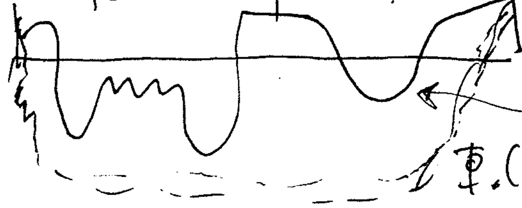
$\Phi_* K$ is now:



$\blacksquare$.

Have: $K$ is neg. somewhere on $\Delta$;
$K'$ anything. $(B_{K'} \# B_K, \eta_{K'} \# \eta_K)$:

$K'$ $\qquad$ $K$



Lemma

$\Phi_*(K' \# K)$.

so can fill w/ genuine contact str.

Rmk: In higher dimensions, ~~how negative~~ Lemma is no longer true, & need to modify to keep track of how negative fns. are, etc. -- less obvious.

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Htpe homotopy equiv. of diff position wrt OT disk.

(previous this is concerned w/ topol. of space of OT disks; open even in dimension 3).