9/15/2015 — J. Ekholm, knot contact homology & topological recursion.

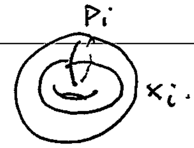
– joint work (in progress) w/ L. Ng & M. Aganagic.

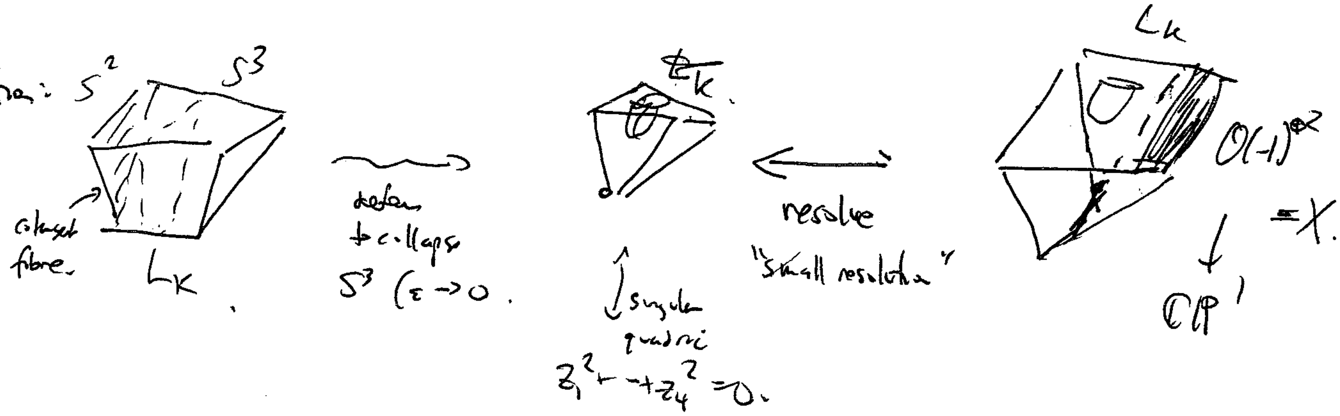
Setup: $K = K_1 \cup \cdots \cup K_m$ in S^3.

L_K conormal of K in T^*S^3

$S^1 \times \mathbb{R}^2$ (if K knot)

$\Lambda_K = L_K \cap U^*S^3$.

$\Lambda_{K_i} \approx$ 

Conifold transition: 

& "push L_K into new space".

e.g., if $L_K \approx S^1 \times \mathbb{R}^2$, push by $d\theta$, it's off of zero section, hence survives to singular quadric.

$c_1(X) = 0$, and Maslov class of $\bar{L}_K$ is 0.

So any hol. curve w/ bd. on $\bar{L}_K$ is rigid.

Want: Count holomorphic curves with boundary on L_K.

$$F = \frac{1}{g_s}\left(F_0 + g_s F_1 + g_s^2 F_2 + \cdots\right)$$

g_s: "string coupling / Planck const." Chern–Simons

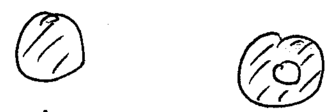
$2 - \chi(\text{curve})$.

$$F_j = \sum_{k_1, \dots, k_m, l} C_{k_1, \dots, k_m, l}\, e^{k_1 x_1} \cdots e^{k_m x_m} Q^l$$

↑ count of curves of with $\chi = -j+1$.

↑ writing homology class of curve in exponential notation

e.g.



-1 0, ---

$$\Psi_K(x_1, \dots, x_m) = \exp(F) \qquad (*).$$

(Conj.) = colored HOMFLY w/ parameters $q = e^{g_s}$, $N = \#$ colors.
$Q = e^{N g_s}$.

Physics proof w/ path integrals.
(Depends on Witten's conjecture
Chern Simons / A-model duality + large N duality,
+ Ooguri-Vafa: $CS(S^3) =$ count of hol. curves in conifold).

(These counts may depend on a framing of the torus ---).
Conj: can pick this to get equality (*).

One can compute the disk potential from knot contact homology.
Knot contact homology associates

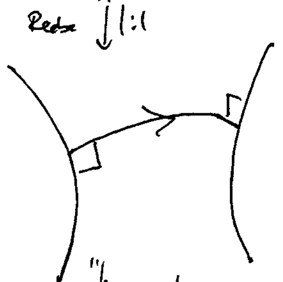
$$\mathcal{A}(\Lambda_K) = \mathbb{Z}[H_2(U^*S^3, \Lambda_K)] \langle \text{Reeb chords} \rangle.$$

Reeb ↕ 1:1

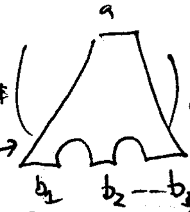
$$= \mathbb{C}[e^{\pm x_i}, e^{\pm p_i}, Q] \langle \text{Reeb chords} \rangle$$

↑ not canonical, b/c rel. homology class   ↖ degree 0, 1, 2.

(∂ has degree -1).

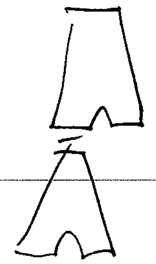


"binormal chords beginning + ending on knot."
index = Morse index

$\partial : \mathcal{A} \longrightarrow \mathcal{A}$ $\quad \partial a = \sum \#\Big($  $\Big) e^{k_1 x_1} \cdots e^{k_m x_m} Q^{l} \cdot b_1 b_2 \cdots b_s$.

$\Lambda_K \times \mathbb{R} \subset U^* S^3 \times \mathbb{R}$.

$\partial^2 = 0$: by SFT cptness.:



$\varepsilon : \mathcal{A} \longrightarrow \mathbb{C}$

$V_K = \{ (e^{x_i}, e^{p_i}, Q) : \exists \ \varepsilon : \mathcal{A} \longrightarrow \mathbb{C} \}$

e.g. $\varepsilon \circ \partial (b) = 0$

b in deg. 1

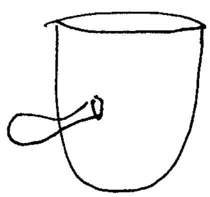
Rmk: It's possible in many cases to calculate explicitly, by count hol curves via combinatorial corresp. & elimination theory!

— Say K knot, so have only e^x, e^p, Q.

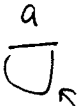
If Parametrize V_K by $p = p(x)$, then $\boxed{p = \frac{\partial F_0}{\partial x}}$ (count of disks).

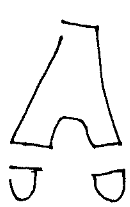
(⇒ disk potential is analytic!)

(point is: V_K is a characteristic variety of a (holonomic) D-module, so is a Lagrangian!).

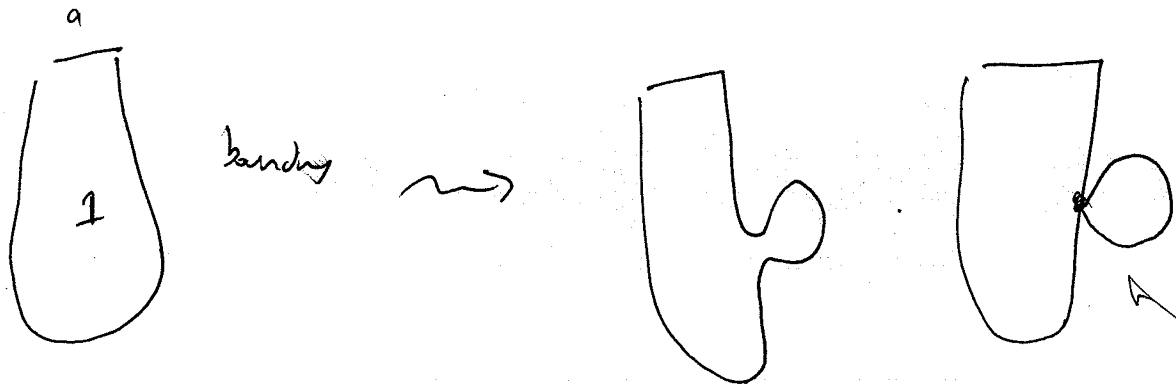


L_K. An exact filling of Λ_K induces an augmentation,

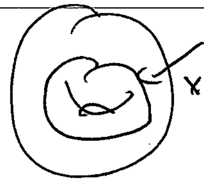
& $\varepsilon(a) = \sum$  ← L_K.

$\varepsilon \circ \partial = 0$:  is a boundary.

L_K non-exact:



Ideal: fix for all disks a bounding chains which goes to ∂ on T^2:

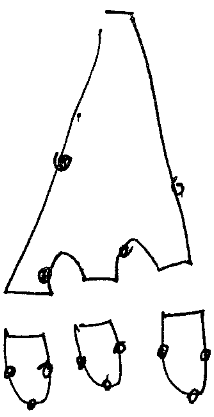


wraps std # of times around generator.

So count also disks w/ insertions ("more pest type bubbles").

surface inside L_K.

pay attention:



$e^{kx}\, e^{mp}$ the number of times can insert

$$p = \frac{\partial F_0}{\partial X}$$

local parametrization along one branch.

effect of insertion:

$$e^{\partial F_0/\partial x} = 1 + \frac{\partial F_0}{\partial x} + \frac{1}{2}\left(\frac{\partial F_0}{\partial X}\right)^2 + \cdots$$

↑ no insertion ↑ one insertion ↑ two insertions, + symmetry of exchanging insertions

Hiding something: ~~there could~~ need to count tree like configurations actually; w/ insertions. (so potential depends on perturbation scheme!

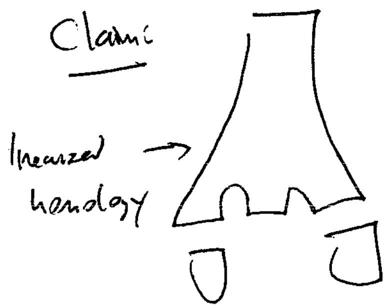
Next, let's try to count annuli.

$$K = K_1 \cup K_2 . \quad L_K = L_{K_1} \cup L_{K_2} .$$

Count annuli w/ one boundary on L_{K_1} & one on L_{K_2}.

(Conj: by physicists: disc + annulus gives recursive computation for everything!
some evidence for torus knots...).

V_K



Claim: for generic augmentations (dense open set),

$$CH^{lin}(K_1, K_2) = 0, \ \&$$

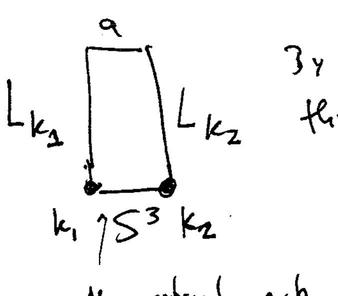
$$CH^{lin}(K_1) = \begin{cases} rk = 1 \text{ in deg. } 1 \ \& \ 2 \\ rk = 0 \text{ in degree } 0. \end{cases}$$

Pf: $L_K \subset T^*S^3$ is now exact.

induces augmentation $p_1 = 0$
$p_2 = 0$.

& can define a map (due to e.g. Cieliebak-Latschev originally)
from chords to chains of paths $K_1 \to K_2$.

By usual action filtration argument ($\exists$ const. strips),
this map is an isomorphism.



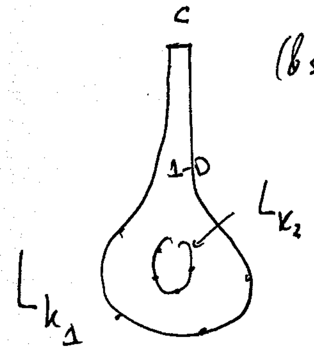
$$\Rightarrow CH^{lin}(\Lambda_{K_1}, \Lambda_{K_2}, \text{ uses } L_{K_1}, L_{K_2})$$
$$\cong$$
$$H_*(P(K_1, K_2))$$ but ~~this is~~ recall: use coeffs. in e^{x_1}, e^{x_2}

e.g. $\partial M_{ax} = (1 - e^{x_1})s_1 + (1 - e^{x_2})s_2$.

so for generic local systems it's actually acyclic (!)

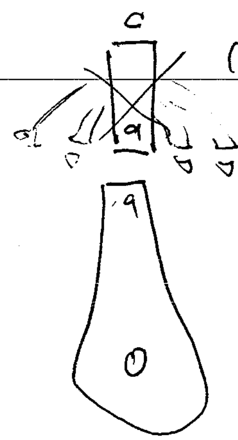
Take a generic (x_1, x_2).

Take a generator of $CH_1^{\text{lin}}(K_1)$, & consider the moduli space



(& secretly used discs everywhere / bounding chains, etc.)

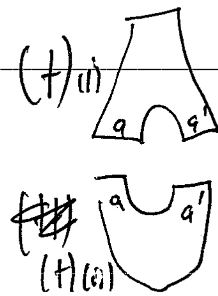
The boundary of this moduli space:



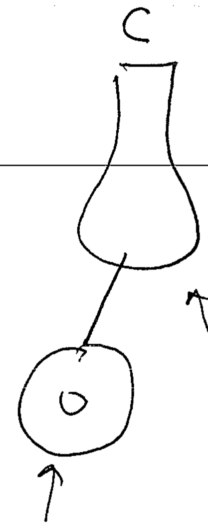
(suppressing augmentations)

This does not contribute, b/c a has no differential;

&



&

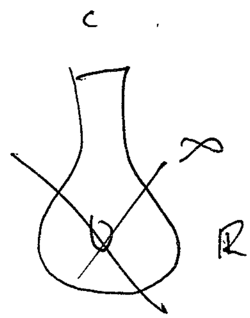


what's this?

this is what we want!

$\Rightarrow$ these two equal.

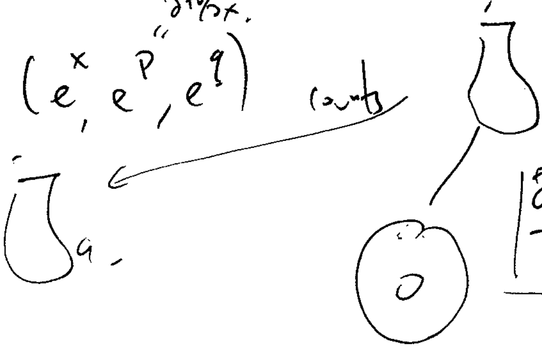
&



$\mathbb{R}$-invt. by geometry.

$\hookrightarrow$ hol. curves starting at c w/ all negative augmentations

$Z_c(e^x, e^p, e^q)$ counts (p ~ $\partial F/\partial x$.)



(ideally O an augmentation)

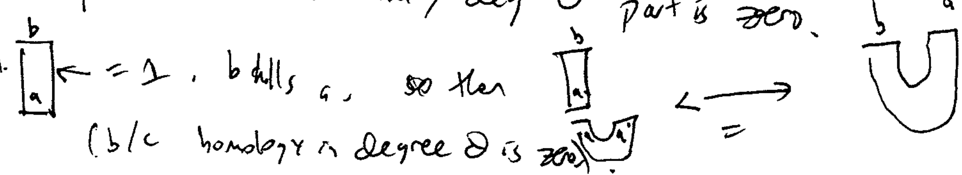
$$\frac{\partial A}{\partial x} \cdot \frac{\partial Z_c}{\partial q}$$

$A(x_1, x_2, Q)$ is the count of annuli. (gen. fxn.)

$= \sum C^{\Theta}_{k_1, k_2, \ell}\, e^{k_1 x} e^{k_2 x_2} Q_2^{\ell}$.

Now to do (+): top piece (i) is part of A differential

Second piece (ii): by lemma, deg O part is zero.



Find b s.t. $\partial b = 1$. b kills a, so then $\longleftrightarrow$ at ∞, so can count these!

(b/c homology in degree 0 is zero) =

So can solve for annulus.

Seems to agree w/ predictions.

In general, try to do same thing:

worst case is when all topology in filling.

By this argument can do more recursion & relate higher complexity things to known computations

⇒ pot'l curves at ∞ determine curves of all genus in interior.

This type of structure can exist outside least contact homology..