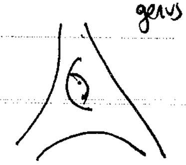
J. Pascaleff, Gluing Fukaya Categories associated to Ribbon graphs.
(Joint w/ Nicolò Sibilla).

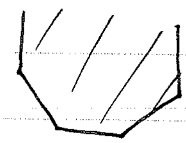
Motivation: Homological Mirror symmetry by gluing local pieces.

e.g., $\Sigma_{g,n}$ (n: holes/punctures; g: genus) $\longleftrightarrow$ toric CY3-fold w/ potential W.



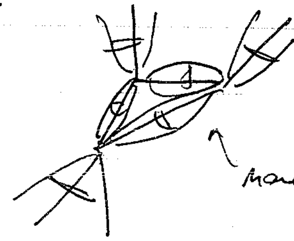
$W(\Sigma_{g,n})$

wrapped Fukaya cat.



W zeros(W) = toric divisor

$D^b\mathrm{Sing}(W^{-1}(0))$
$\shortparallel$
$MF(W)$.



moment map image of crit locus

cf. $\boxed{\text{Heather Lee's thesis}}$.

Strategy: Break $\Sigma_{g,n}$ into pairs of pants & break
$MF(W)$ into $MF($  $)$ ✓

$W($  $) \simeq MF(\mathbb{C}^3, xyz)$.

want to say our categories are sheaves of (dg/A_∞/∞/ ...) category.

Sheaf on what?

e.g.,: MF^∞ is a sheaf in the étale topology. (Preygel)

Tabuada: ∃ 2 "model structures" on small dg categories
1) weak eq. are ~. equiv.
2) weak eq. are Morita equiv.

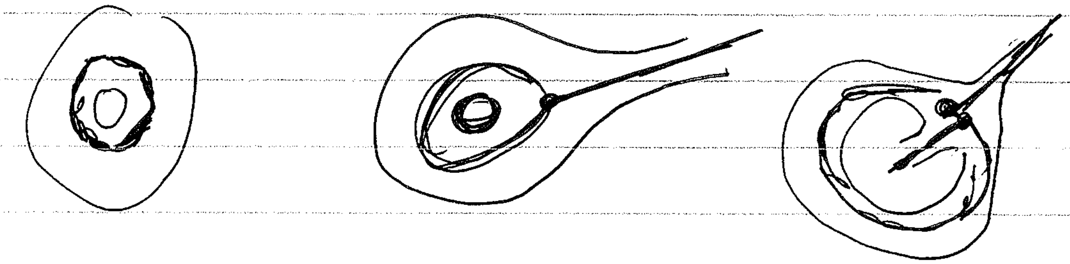
$U \subset V \rightsquigarrow \underset{\text{cat}}{\mathcal{F}(V)} \xrightarrow[\mathrm{res}_{V,U}]{\text{functor}} \underset{\text{cat}}{\mathcal{F}(U)}$, +

"fiber product."

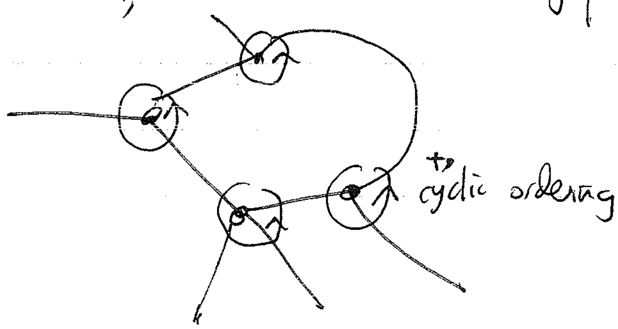
$$\begin{tikzcd} \mathcal{F}(U \cup V) \arrow[r] \arrow[d] & \mathcal{F}(U) \arrow[d] \\ \mathcal{F}(V) \arrow[r] & \mathcal{F}(U \cap V) \end{tikzcd}$$

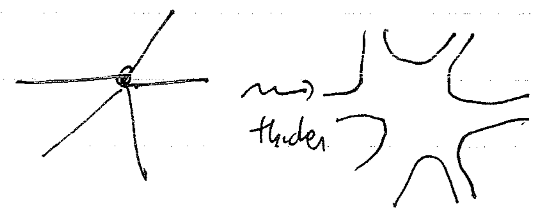
~~Symplectic~~ Fukaya categories are (suppose) to be) co-sheaves on the skeleton of manifold

Ex:



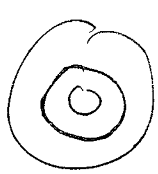
In dim 2, skeleton is a Ribbon graph,





Two variants: Fukaya ⇝ compact Lags (retract onto skeleton).
co Fukaya ⇝ partially wrapped Fukaya.

E.g. annulus:



coFuk $= \mathcal{W}($  $) =$ f.g. $k[x,x^{-1}]$-modules

Fuk $= \mathcal{F}_{cpct}($  $) =$ f.d.$/\mathbb{k}$ $k(x,x^{-1}]$-modules

Dyckerhoff – Kapranov's topological model of the co) Fukaya category:

Connes' Cyclic category Λ: objects $\langle n\rangle = \left\{ \exp \frac{2\pi i k}{n+1} \right\} \subseteq \mathbb{C}$

morphisms: $[f]: S^1 \longrightarrow S^1$ mapping $\langle n\rangle$ to $\langle m\rangle$.

$\langle n\rangle \to \langle m\rangle$ are htopy classes. $f(\langle n\rangle) \subseteq \langle m\rangle$.

cyclic object $\underset{\wedge}{\overset{\text{in } \mathcal{C}}{}}$: functor $\underline{\Lambda}^{op} \longrightarrow \mathcal{C}$.

co-cyclic object - - - $\underline{\Lambda} \longrightarrow \mathcal{C}$

<u>Def</u>: $\mathcal{E}^n = D^{(2)}(A_n\text{-mod})$ ← derived category of modules, $\mathbb{Z}_2$ graded (b/c of (2)).

$\bullet \to \bullet \to \bullet \cdots \to \bullet$

$\mathcal{E}^\bullet : \underline{\Lambda} \longrightarrow dgcat^{(2)}$ ← dg cat. of $\mathbb{Z}_2$ graded dg categories

is a cocyclic object.

(not obvious, ~~etc~~ e.g. $\Rightarrow D^{(2)}(A_n\text{-mod})$)

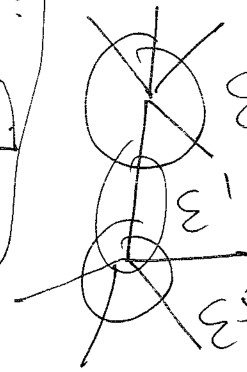
Ribbon graph $\Gamma \longrightarrow \underline{\Lambda}^{\Gamma}$ ~~$\mathcal{U}_\Gamma$~~ $\in \mathrm{Set}_{\underline{\Lambda}}$ ← cyclic objects ~~in sets~~.

$$\mathrm{coFuk}(\Gamma) := \underset{\to \{\Lambda^n \to \Lambda^\Gamma\}}{\overset{(\text{hocolim?})}{\mathrm{holim}}} \mathcal{E}^n$$

[D. K.]

<u>Thm</u>: This is independent of choice of ribbon graph, only depends on top. type of Ribbon graph.

$\boxed{\text{Ex}}$:



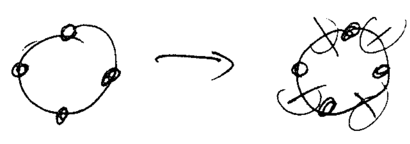
$$\mathcal{E}^1 \nearrow \mathcal{E}^4 \searrow \mathrm{coFuk}, \quad \mathcal{E}^1 \searrow \mathcal{E}^4 \nearrow \mathrm{coFuk}$$

$\mathcal{E}_\bullet : \Lambda^{op} \longrightarrow dgcat^{(2)}$

$\langle n \rangle \longrightarrow (\mathcal{E}^{\langle n \rangle *})^{op}$

$\uparrow$

$* : \underline{\Lambda} \to \underline{\Lambda}^{op}$



maps go other way, & take op at the end.

$$\mathrm{Fuk}(\Gamma) = \underset{\leftarrow \{\Lambda^n \to \Lambda^\Gamma\}}{\mathrm{holim}} \mathcal{E}_n.$$

coFuk(Γ), Fuk(Γ)

Thm (Dyckerhoff - Kapranov): Only depends on marked surface.
(~~follows from extra~~ property of Σ°: it's a "2-segal object").

Rmk: $\mathcal{F}$ cosheaf

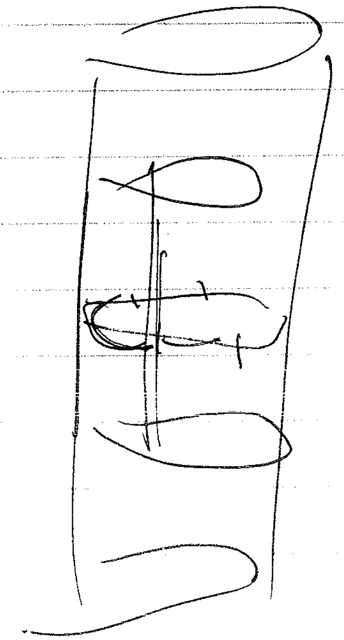
$U \subset V$

$\mathcal{F}(U) \to \mathcal{F}(V)$,

$$\begin{tikzcd} \mathcal{F}(U \cup V) & \mathcal{F}(U) \arrow[l] \\ \mathcal{F}(V) \arrow[u] & \mathcal{F}(U \cap V) \arrow[l] \arrow[u] \end{tikzcd}$$

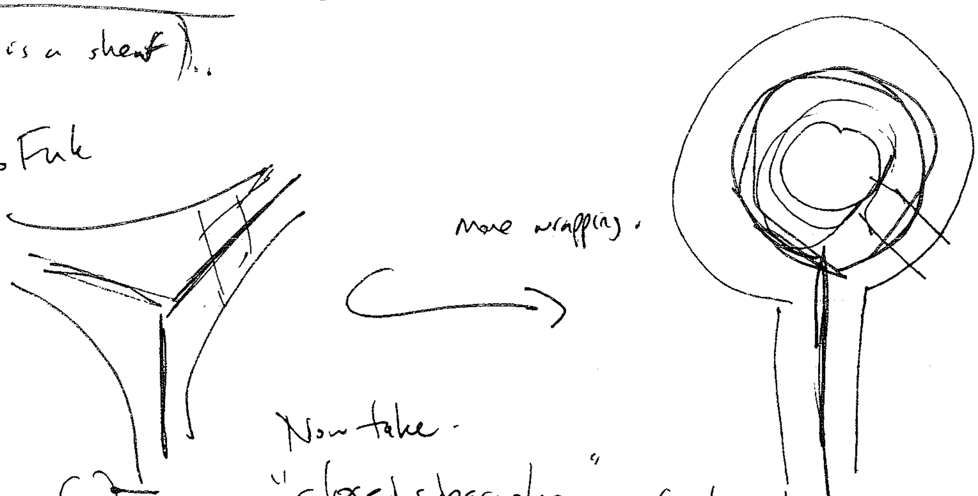
is "(htpy) pushout" of sections

as in Seifert - Van Kampen theorem.



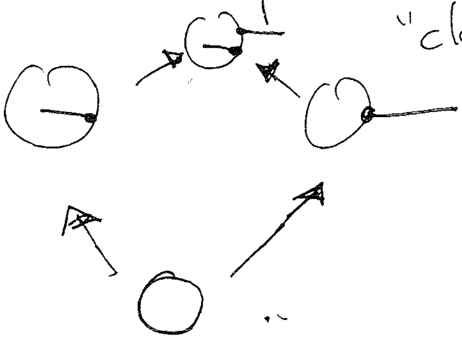
coFuk(Γ) is a cosheaf.
(& Fuk is a sheaf)..

Ex: coFuk



Now take "closed subgraphs." (not just open subgraphs)

coFuk



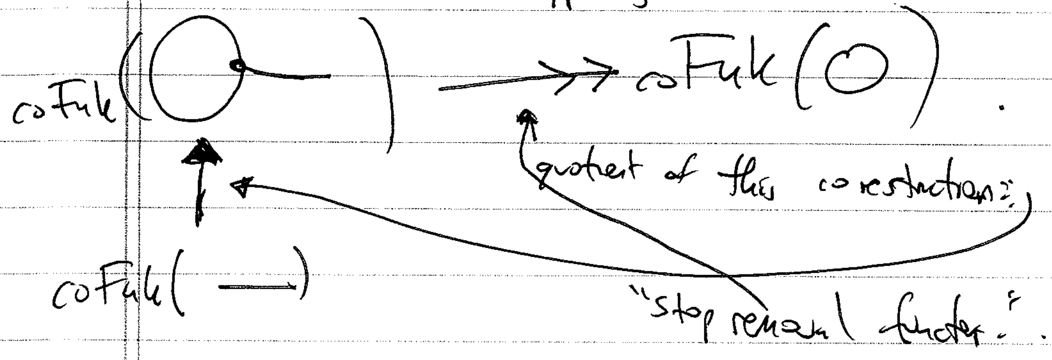
co~~Fuk~~

~~Perf($\mathbb{C}^*$)~~

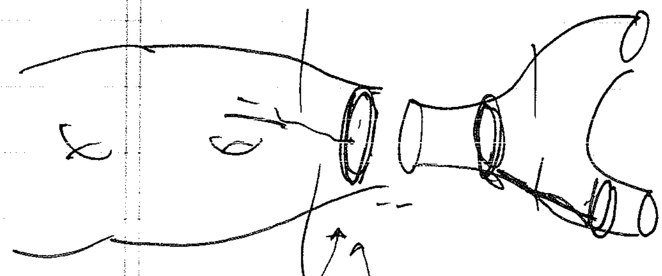
$$\begin{tikzcd} & \mathrm{Perf}(\mathbb{P}^1) \arrow[dl, shift left] \arrow[dr, shift left] & \\ \mathrm{Perf}(\mathbb{C}) \arrow[ur, shift left] \arrow[dr, shift left] & & \mathrm{Perf}(\mathbb{C}) \arrow[ul, shift left] \arrow[dl, shift left] \\ & \mathrm{Perf}(\mathbb{C}^*) \arrow[ul, shift left] \arrow[ur, shift left] & \end{tikzcd}$$

This is a sheaf in this case.

How is the restriction functor happening?



Idea: this is the type of gluing we want to do.



Check locally categories are gluing, but cosheaf ~~is the~~ ~~...~~ property doesn't help.

cat. theory tricks to turn "colimits into limits."

Fukaya is "Matlis dual" of coFukaya.

$$\mathrm{Fuk} = \underline{\mathrm{RHom}}(\mathrm{coFuk}, k)$$

but cannot switch