Talbot Day I Talk I - Rui,
Lag'n submfolds.

(M,ω) 2n-sympl mfold.

ω non-deg. closed 2-form on M.

$[\omega] \in H^2(M;\mathbb{R})$

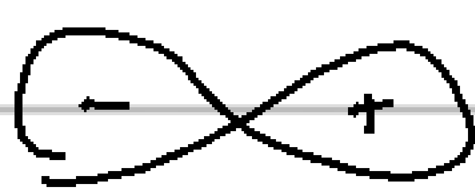
L n-dim'l mfold.

$i: L \to M$, $i^*\omega = 0$ gives us an immersed or embedded Lagrangian submfold (if i immersion, embedding).

$\omega = -d\alpha$, $i^*\alpha = dH$ means an exact Lag'n.

Gromov: $(\mathbb{C}^n, \omega_0)$
embedded compact $L \hookrightarrow \mathbb{C}^n$
$\Rightarrow [\alpha] \neq 0$.

Ex. of exact Lag'n: For an immersed curve in $\mathbb{C}$,

e.g. 
, exact Lag'n $\Leftrightarrow$ area of "+" = area of "−" by Stokes.

Lag. isotopy:

$\varphi: [0,1]\times L \to M$, $\varphi^*\omega = dt\wedge\alpha + \beta$

$j_t : L \hookrightarrow [0,1]\times L$. ($d\beta = 0 \iff j_t^*\alpha$ closed)

$j_t^*\alpha$ closed 1-form. $\forall t$.

exact Lag'n isotopy if $j_t^*\alpha$ exact, $j_t^*\alpha = dH$.

Hamiltonian isotopy ..

may depend on t.

(M, L)

$w : (D^2, \partial D^2) \longrightarrow (M, L)$ smooth.

$I_\omega(w) = \int_{D^2} w^*\omega$ $\boxed{\text{symplectic area}}$

* This is invariant only under exact Lag'n isotopy

Disk D^2 bounding L.

- -

Maslov class ; i.e. relative 1st Chern class

$$[\mu] \in H^2(M, L; \mathbb{Z})$$

$$u : (D^2, \partial D^2) \longrightarrow (M, L)$$

$$\begin{tikzcd}
(u^* TM, u^* TL) \arrow[d] & (TM, TL) \arrow[d] \\
(D^2, \partial D^2) \arrow[r, "u"] & (M, L)
\end{tikzcd}$$

always has a symplectic trivialization, i.e. can find a map

$$\begin{tikzcd}
(\mathbb{C}^n, \mathbb{R}^n) \arrow[r] \arrow[dr] & (u^* TM, u^* TL) \arrow[d] \\
& (D^2, \partial D^2)
\end{tikzcd}$$

Naturally get a loop

$\gamma = \Phi^* u|_{\partial D^2}$ in Lag'n Grassmannian. $\Lambda(n)$

$$\Lambda(n) = \{ A \cdot \mathbb{R}^n \mid A \in U(n) \} / \sim$$

$$A_1 \mathbb{R}^n = A_2 \mathbb{R}^n \iff A_1 A_1^t = A_2 A_2^t$$

Maslov index $\mu(\gamma) = \deg(\det A \cdot A^t)$

This translation is unique up to homotopy, & $\mu(\gamma)$ indep. of htpy.

Again, this AA^t is on $\Lambda(n) := U(n)/O(n)$ as a homology class.

$I_\mu([w])$ well defined (Maslov class), i.e.

Get a homomorphism $\mu : \pi_2(M, L) \to \mathbb{Z}$.

__Prop:__ $w : (D^2, \partial D^2) \longrightarrow (M, L)$

$\bar{w} : (D^2, \partial D^2) \longrightarrow (M, L)$

$w\big|_{\partial D^2} = \bar{w}\big|_{\partial D^2}$

Get, by gluing, $u : S^2 \longrightarrow M$, $u = \begin{cases} w(z) & z \in D^2 \\ \bar{w}(z) & z \in -D^2 \end{cases}$

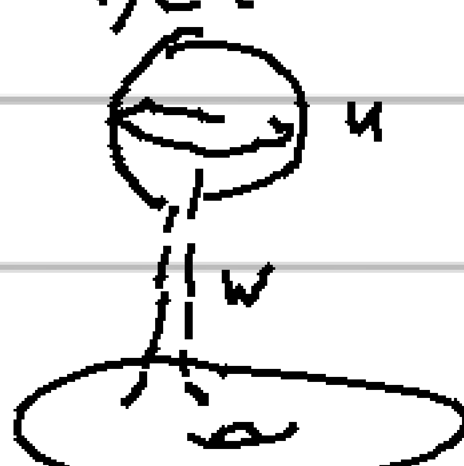
$\qquad\qquad \overset{\shortparallel}{D^2 \#_{\partial D^2} (D^2)}$

__Claim:__

$$I_{\mu,L}(w) - I_{\mu,L}(\bar{w}) = 2\, \overset{c_1(TM)([u])}{c_1 \cdot [u]}$$

(proof omitted).

Using this property:



Monotone Lag'n submanifold :

(M, L)

$$I_\omega = \lambda I_{M,L} \; , \; \lambda \geq 0 \text{ const.}$$

(NB: only invariant under exact Lag'n isotopy.)

For this sphere, if it's monotone, then

$$I_\omega = 2\lambda I_{C_1}, \text{ i.e.}$$

the symplectic manifold needs to be monotone as well for the Lag'n to be monotone

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Darboux-Weinstein theory :

$L \hookrightarrow M$ immersed (embedded) Lag'n in (M, ω)

T^*L

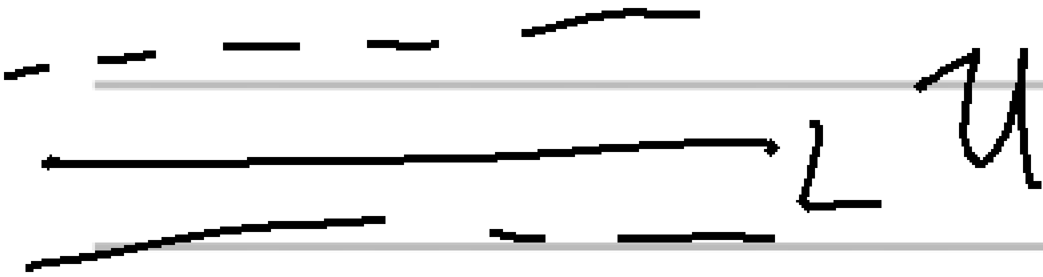
$\downarrow$

L

$\omega_{can} = -d\alpha_{can}$

$\theta \cdot \quad \theta^*\alpha_{can} = 0 \;\; \forall \theta \in \Omega^1(L)$

Darboux-Weinstein thm:



$\Phi : U \longrightarrow M$ immersed/embedding

$\Phi^* \omega = \omega_{can}$.

$i \circ \Phi = z$ ← zero section.

U = nbhd of zero section of L in T^*L.

Second example:

(1)

(2) anti-symplectic involution.

(M, ω)

$\sigma : M \longrightarrow M$. $\begin{cases} \sigma^* \omega = -\omega \\ \sigma^2 = Id \end{cases}$

$L = Fix(\sigma)$

ex. $\mathbb{RP}^n \subseteq \mathbb{CP}^n$.

Prop:

If (M,ω) monotone (i.e. $I_\omega = 2\lambda F_{c_1}$), then $\Rightarrow L = \mathrm{Fix}(\sigma)$ monotone Lag'n submfld.

(3) Clifford torus in $\mathbb{C}P^n$.

$$T^{n+1} = \underbrace{S^1(1)\times S^1(1)\times \cdots \times S^1(1)}_{n+1} \hookrightarrow S^{2n+1}(1) \subset \mathbb{C}^n$$

isometric embedding.

$$T^{n+1}/S^1 \text{ action} := T^n.\ \text{Clifford.}$$

Claim: monotone Lag'n submfld in $\mathbb{C}P^n$.

Proof: $\to \cancel{\pi_2(T^n)} \to \pi_2(\mathbb{C}P^n) \to \pi_2(\mathbb{C}P^n, T^n)$

$\to \pi_1(T^n) \to \pi_1(\mathbb{C}P^n) \to \cdots$

$$\pi_2(\mathbb{C}P^n, T^n) \cong \pi_2(\mathbb{C}P^n) \oplus \pi_1(T^n)$$

$$w_k(z) = [1 : 1 : \cdots : 1 : z : 1 : \cdots : 1]$$

n generators w_k $k=1,\dots,n$.

$I_\mu(w_k) = 2.$

$I_\mu(w_0) = 2.$

Matrix in $\Lambda(n)$ can be written as

$$\omega\left(\begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & e^{2\pi i t} & & \\ & & & \ddots & \\ 0 & & & & 1 \end{bmatrix}\right) \in U(n)$$

is defined as 2.

$$\pi_2(\mathbb{C}P^n) \xrightarrow{i} \pi_2(\mathbb{C}P^n, T^n) \xrightarrow{j} \pi_1(T^n)$$

$$j([w_0] + \cdots + [w_n]) = 0$$

$$\overset{\shortparallel}{i([\alpha])}$$

so $I_\mu(j([w_0] + \cdots + [w_n])) = 2(n+1)$.

$2(n+1)$

i.e. $[w_0], \ldots, [w_n]$ give us the generators in $\pi_2(\mathbb{C}P^n, T^n)$.

$I_\omega([\alpha]) = \pi$.

$2I_c([\alpha]) = 2(n+1)$.

$$\lambda = \frac{\pi}{2(n+1)}$$

(4) Standard torus in $\mathbb{C}^n$.

$$T^n_{a_1,\ldots,a_n} = S^1(a_1)\times \cdots \times S^1(a_n) \hookrightarrow \mathbb{C}^n$$

monotone $\Leftrightarrow a_1 = \cdots = a_n$, monotone.

monotonicity constant $\lambda = \pi \cdot a^2/2$ (?)

(5) Chekanov torus

$S^1 := \mathbb{R}/\mathbb{Z}$

$$S^1\times\mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$$

$$(t, x_1, \ldots, x_n) \longmapsto (e^{x_1}\cos 2\pi t, e^{x_1}\sin 2\pi t, x_2, \ldots, x_n)$$

$$(i^*)^{-1} : T^*(S^1\times\mathbb{R}^n) \longrightarrow T^*\mathbb{R}^{n+1} \cong \mathbb{C}^{n+1}$$

Given

L is embedded Lag'n in $\mathbb{C}^n$, get:

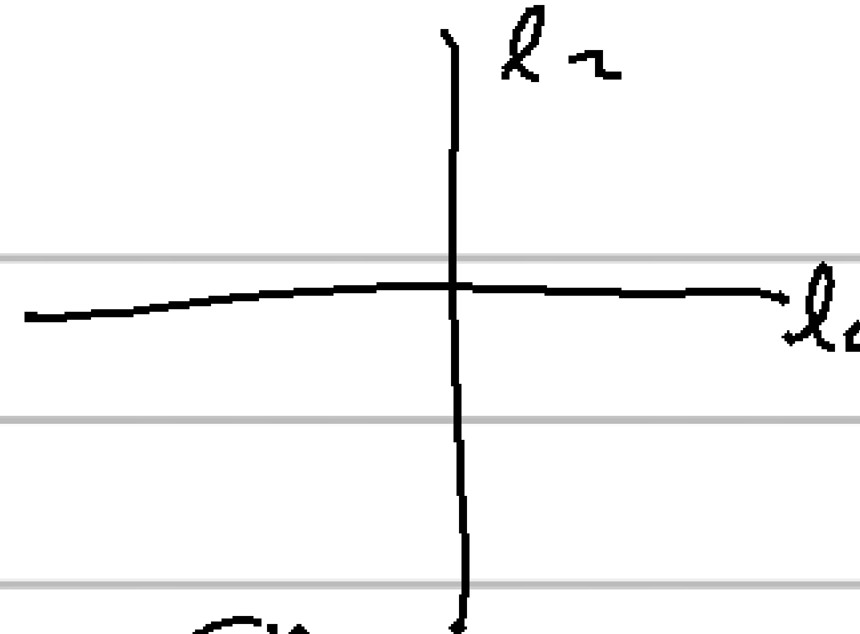
$$\Theta_a(L) = (i^*)^{-1}(\overset{(\text{or } a)}{\text{zero section}}(S^1)\times L)$$

embedded Lag'n in $\mathbb{C}^{n+1}$, called <u>Chekanov torus</u>

When L is <u>monotone</u>, $\Theta_0(L)$ is <u>monotone</u>.

Lag. surgery

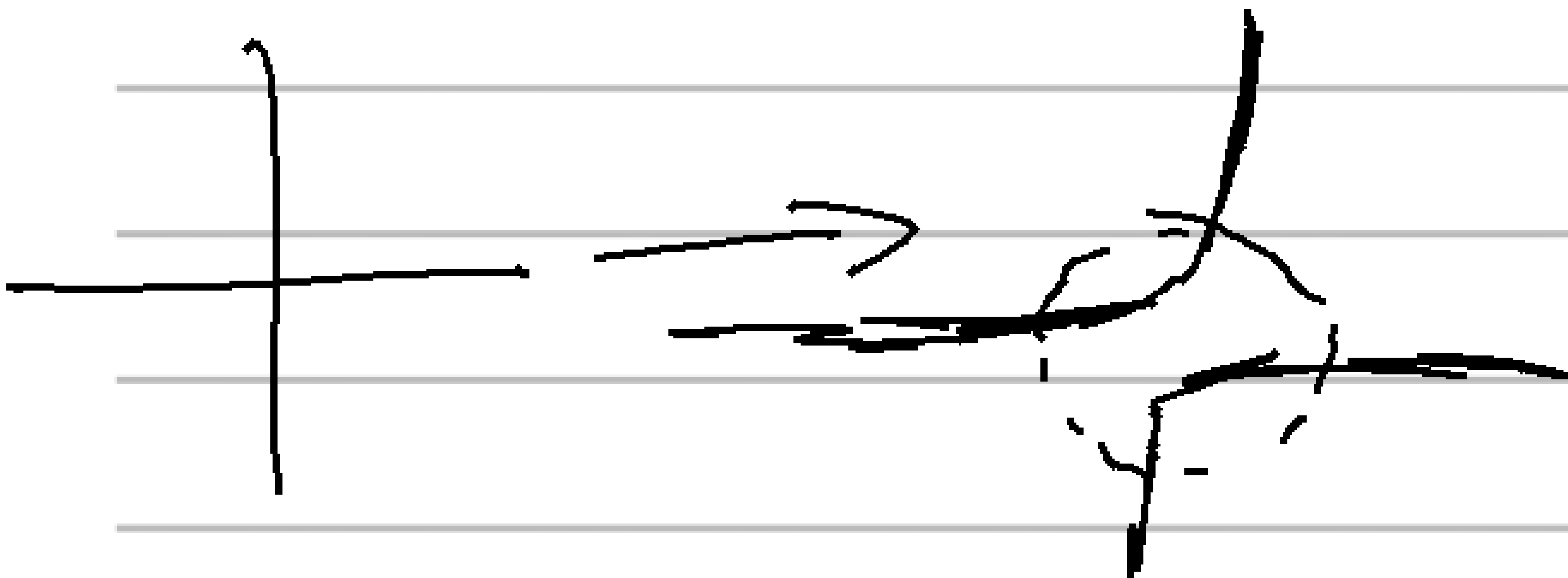
(V^{2n}, ω)



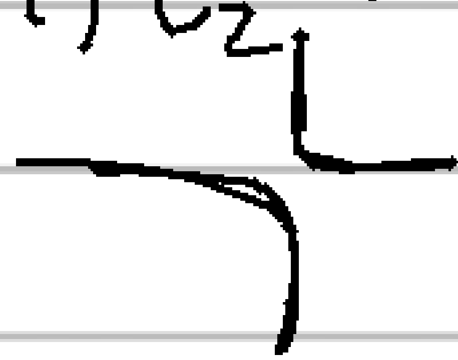
$f: S^{n-1} \times \mathbb{R} \hookrightarrow \mathbb{C}^n$

$f(S^{n-1} \times [c, +\infty)) = l_1 \setminus B_1$ → disc in l_1.

$f(S^{n-1} \times (-\infty, -c]) = l_2 \setminus B_2$ → disc in l_2.



order of l_1, l_2 important here.

If did  , get non-homotopic guys,

And much, much more.