Talbot Day 1 Talk 2 : Fan, Formal aspects of Floer theory

<u>Morse theory</u>:

$f : M \to \mathbb{R}$, $\dim M = n$, $df = 0$. crit. pt

near $p \in$ crit, $-x_1^2 - x_2^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2$.

p is of <u>index</u> i

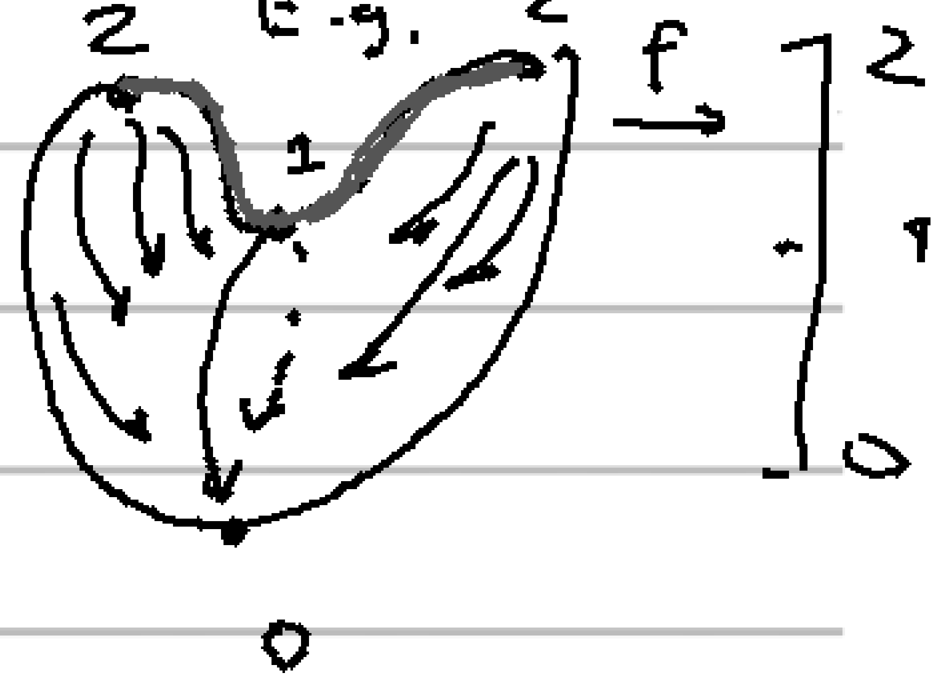
Choose g on M

$\dot{u}(t) = -\nabla f(u(t))$

$u : \mathbb{R} \to M$.

φ_s is the flow

E.g.



<u>stable mfold</u> :

$W^s(x) = \{ z \in M \mid \lim_{s \to \infty} \varphi^s(z) = x \}$ ← dim $n-i$

<u>unstable mfold</u>:

$W^u(x) = \{ \text{"} \quad {}_{s \to -\infty} \quad \text{"} \}$ ← dim i

Given x, y crit. points,

$\mathcal{M}(x,y) = W^s(y) \cap W^u(x)$

If f is <u>Morse-Smale</u>, this intersection is transverse and has the right dimension, namely $\operatorname{ind}(x) - \operatorname{ind}(y)$.

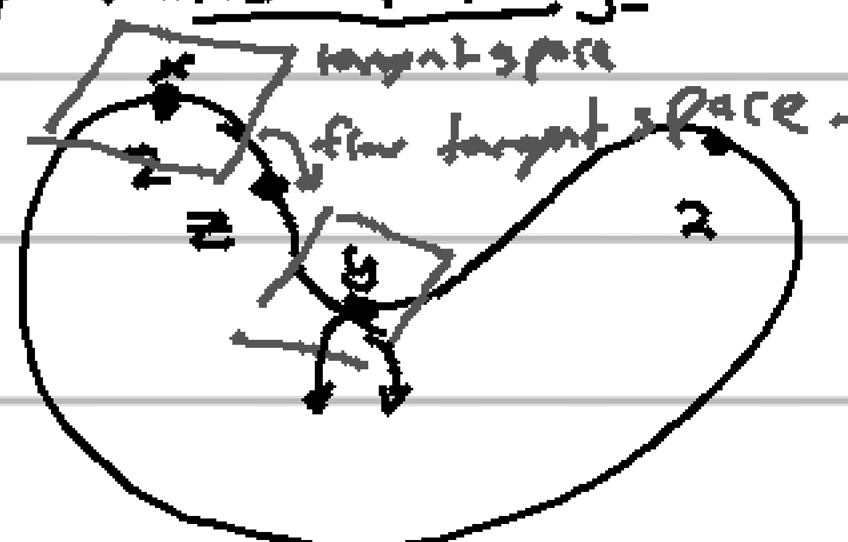
Define $\check{\mathcal{M}}(x,y) = \mathcal{M}(x,y)/\mathbb{R}$, moduli space of flowlines from x to y.

For most f, this will be a smooth manifold.

From this, we'll define the <u>Morse complex</u> CM_k.

$$CM_k = \bigoplus_{\mathrm{ind}(p)=k} \mathbb{Z}\langle p\rangle.$$

concept of <u>stable framing</u>.



For any $z \in \mathcal{M}(x,y)$,

$$T_z W^u(x) / \text{direction of flowline} \cong T_x W^u(x)$$

(b/c W^u is a disc).

map uniquely defined up to htpy

Normal bdle $\nu_z W^u(x) \cong T_y W^u(y)$

$$T_z \mathcal{M}(x,y) = T_z W^u(x) \cap T_z W^s(y) = \ker\left(T_z W^u(x) \to \nu_z W^s(y)\right)$$

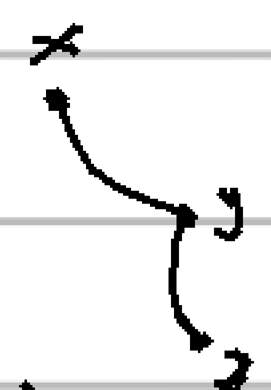
$$= \underbrace{\ker\left(T_x W^u(x) \xrightarrow[\substack{\text{map,}\\ \text{depends}\\ \text{on } z}]{\text{surjective}} T_y W^u(y)\right)}_{\text{trivial bdles.}}$$

So $T_z\mathcal{M} \oplus T_y W^u(y) \cong T_x W^u(x)$. $(*)$

$$\partial \underbrace{\langle x \rangle}_{\text{ind. } k} = \sum_{\substack{y \text{ of ind } k-1\\ \text{flow from } x \to y}} \varepsilon \langle y \rangle.$$

What is ε? Choose an orientation for each $W^u(x)$, and use $(*)$ to determine $+$ or $-$ signs based on compatibility of splitting w/ orientations of $T_p W^u(p)$'s.

$\mathcal{M}(x,y)$.

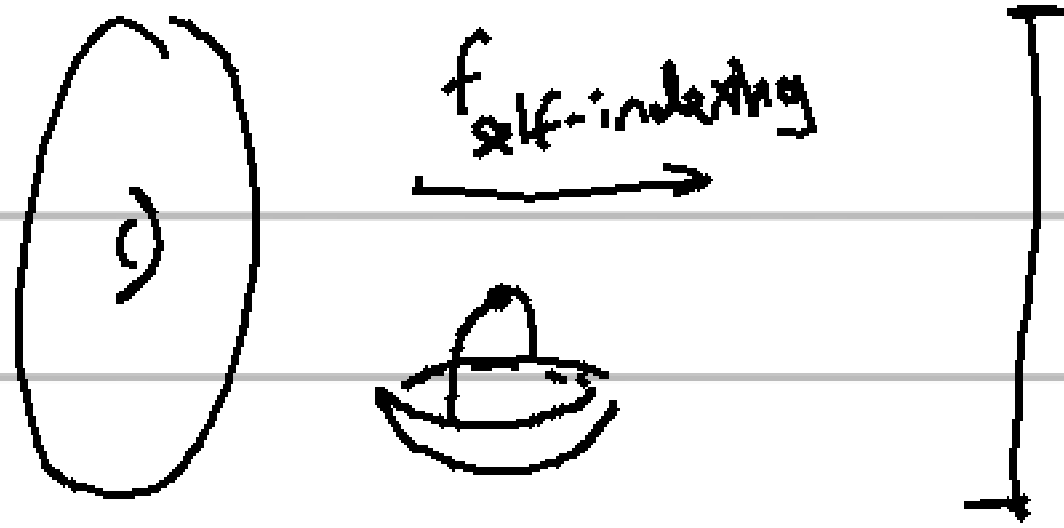


Why is $\partial^2 = 0$? ∂^2 is the sum of

Look at $\mathcal{M}(x,z)$: a 1-dim'l mfold,

so boundary points occur in pairs.

And need to show $\partial \hat{\mathcal{M}}(x,z) = $ (piecewise flows ).



$f^{-1}([-\infty, k]) = X_k$.

X_k / X_{k-1} looks like a bouquet of spheres.

From this, can recover stable htpy type of mfold.

Four things: 1) $\mathbb{Z}$-graded complex
2) $\mathbb{Z}$-coeffs
3) captures topology
4) nice moduli space of flows

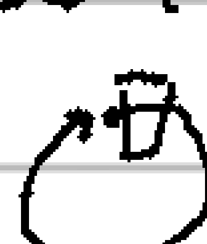
All four won't hold in Floer homology

In Morse theory, started with $f: M \to \mathbb{R}$, got

$df \rightsquigarrow \nabla f$

instead of df, use α a closed 1-form.

Things that could go wrong:



may have orbits
want non-degeneracy, i.e.,
tangent space doesn't come
back to identity.

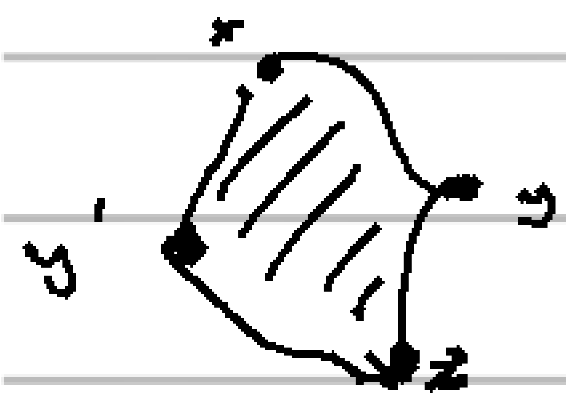
<u>unknown</u>: go to cover of M? need to recompute action of covering.

We'll introduce the Novikov ring

$$\left\{ \sum_{\substack{a_i \in \mathbb{Z} \\ r_i \to \infty}} a_i t^{r_i} \right\} = N$$

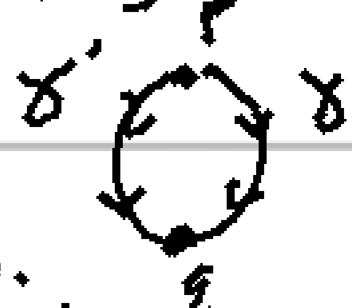
i.e. only fin. many r_i below any K.

Now, $$\partial \langle x \rangle = \sum_{x \to y} \varepsilon\, t^{\int \alpha} \langle y \rangle$$



Again, $\partial^2 = 0$ by same argument. Point is that weights of t for two paths like above are the same b/c α is closed by Stokes

<u>Examples</u>: $d\theta$ on S^1



$$\int_\gamma \alpha \neq \int_{\gamma'} \alpha$$

$N \xrightarrow{\partial \neq 0} N$

$$\partial \langle p \rangle = (t^{c_1} - t^{c_2}) \langle q \rangle$$

Def:
E is a polarized Hilbert space if it has
$\{A \mid A^2 = I + \text{cpct.}\}$
$E \underset{\text{approx}}{\approx} E_+ \oplus E_-$, up to some finite dimensional stuff

A polarization is a choice of J mod cpct. choices.

ex: $\mathcal{L}M$
$\downarrow$
γ

Consider $H =$
$\{\text{vector fields along } \gamma \in T_\gamma \mathcal{L}M\} \ni X$

$X \longmapsto J \cdot \frac{DX}{D\theta}$, J an a.c.s., θ parameter of γ.

covariant diff (choose g)

(make sure actual J via spectral projection)

this doesn't square to -1, but is a polarization.

$GL_{res}(E) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;\middle|\; \begin{matrix} A, D \text{ Fredholm} \\ B, C \text{ compact} \end{matrix} \right\}$

restricted $\big|$ 2 htpc

$\mathbb{Z} \times BO$ (classifying space for vec. bdles).

M is a polarized Hilbert m'fold if $T_x M$ all have polarizations. (smoothly varying or something), (i.e. given by structure group GL_{res}).

Get a $\overset{\text{classifying}}{\text{map}}$

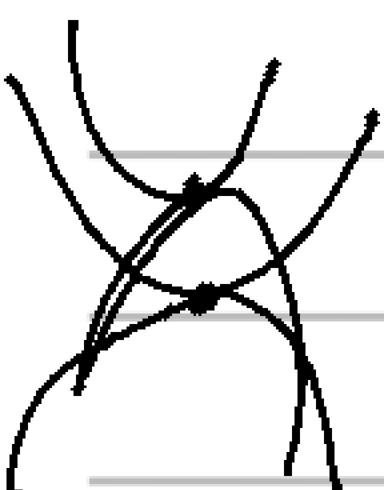
$$M \longrightarrow BGL_{\text{res}} = U(\infty)/O(\infty) = Gr^{\text{Lag}}_{\infty, 2\cdot\infty}$$

$$\pi_1\left(U(\infty)/O(\infty)\right) = \mathbb{Z}.$$

Think of the polarization as giving us stable & unstable mfolds.

$$\pi_1(M) \longrightarrow \pi_1\left(U(\infty)/O(\infty)\right)$$

Index is only well-defined up to shifting by above map.



"spectral flow"

(pos. things flowing to negative things).

Stable framings are obstructed here, by

$$H^2(U(\infty)/O(\infty))$$

Stable framing gave us signs in Morse complex

related to consistent orientation of moduli spaces of flows.

we won't always have this. (forces us to use $\mathbb{Z}/2$ coeffs.)

$$\Omega^{(M,\omega)}(L_0, L_1) = \{ u \in C^\infty([0,1], M) \mid u(0) \in L_0,\ u(1) \in L_1 \}$$

X a v.f. along u

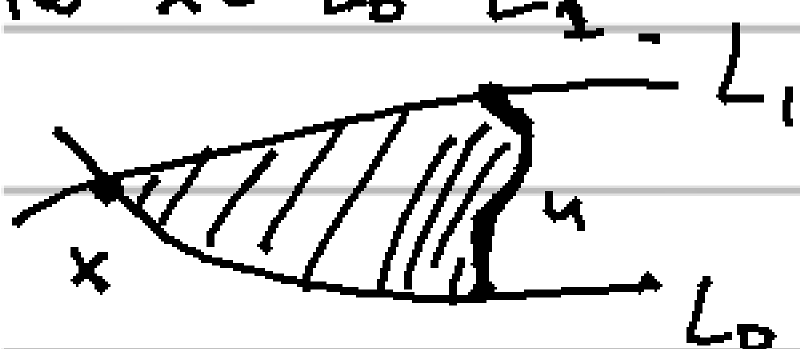
$\cap$

$T_u \Omega(L_0, L_1)$

$$\alpha(X) = \int_0^1 \omega(\dot{u}(t), X)\, dt$$

α "=" dA, where

for $x \in L_0 \cap L_1$.



$$A(u) = \int \phi^* \omega.$$

$\phi : \mathbb{R} \times [0,1] \to M$