Day 1, talk 3 Kevin: A_∞ structures

dgas

Def: A diff. graded algebra (dga) is
$A^\bullet$ graded alg. d differential, s.t.
$d(ab) = da \cdot b + (-1)^a db$

Cohomology is an algebra also!

Problem: quasi-isos are not necessarily invertible!

ex: k = field

$k[\alpha]/\alpha^2$, $d = 0$
deg 2.
cohomology = $k \oplus 0 \oplus k$

$\exists$ quasi-iso this way ($\beta_2 \mapsto \alpha$) but not other way! Cause needs to $\mapsto \beta_1$.

deg 2 deg 2 deg 1
$k[\beta_1, \beta_2, \beta_3]/\beta_1^2$

$d\beta_1 = 0$
$d\beta_2 = 0$
$d\beta_3 = \beta_1$

cohomology is:
$k \oplus 0 \oplus k \oplus H^3 \oplus \ldots$
1 β_2

kill H^3 by adding vars of degree 2.
kill H^4 " " deg. 3.

$\downarrow$ $\simeq$
$k[\beta_1, \beta_2, \ldots]/\beta_1^2$
$\downarrow$ homology
$k \oplus 0 \oplus k$

__example__: rational homotopy theory

study rat'l htpy type of a space X by studying $C^*(X;\mathbb{Q})$ w/ cup product.

__Theorem__: (roughly)

X, Y have the same rat'l htopy type iff there is a chain of q-isoms

$$C^*(X;\mathbb{Q}) \to D_1 \leftarrow D_2 \to \cdots \leftarrow D_n \to C^*(Y;\mathbb{Q})$$

__Fix__: A_∞-algebras $\supset$ dga (subcat, but not full)

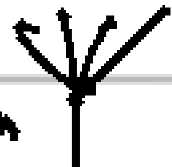
(__Paul__: the map singular cohom $\longrightarrow$ de Rham cohom was shown to be a quasi-isom of dgas by constructing an A_∞ morphism, to correct for non-commut. of cup product on sing. cohomology side).

__A_∞-algebra__:

__Def 1__: Given by following data

(1) V^* $\mathbb{Z}$-graded v.sp.

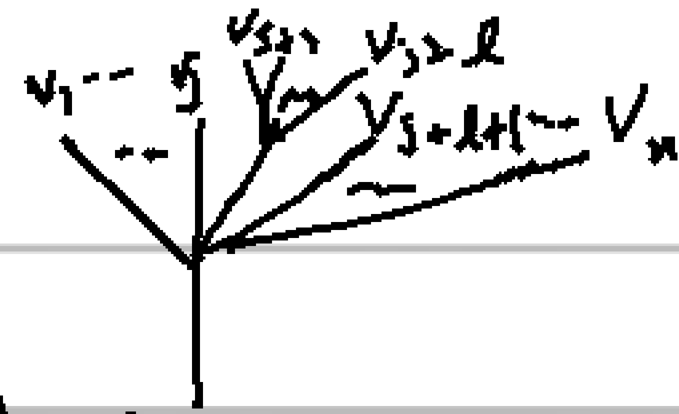
(2) operations

m_n 

$$m_n : V^{\otimes n} \longrightarrow V[2-n], \quad n \geq 1.$$

(3) satisfying:

$$\sum_{k+l=n+1}\ \sum_{j=0}^{k-1}(-1)^{*}$$



$$= 0.$$

$n=1$: $m_1(m_1(v_1))=0$

$d := m_1 \implies d$ has deg 1, $d^2=0$.

<u>$n=2$:</u>

$k=1, l=2$: $\longrightarrow m_1(m_2(v_1,v_2))$

$k=2, l=1$: $\longrightarrow m_2(m_1(v_1),v_2) \pm m_2(v_1, m_1(v_2))$

Denoting $a\cdot b := m_2(a,b)$, this is the Leibniz rule.

<u>$n=3$</u>: $v_1\cdot(v_2\cdot v_3) - (v_1\cdot v_2)\cdot v_3 =$

$$d m_3(v_1,v_2,v_3) \pm m_3(dv_1,v_2,v_3)$$
$$\pm\ m_3(v_1,dv_2,v_3) \pm m_3(v_1,v_2,dv_3).$$

i.e. this implies associativity on homology ("associativity up to htpy")

<u>example</u>: a dga is an A_∞ alg. w/ $m_n=0$, $n \geq 3$.

<u>Def 2</u>: following data: (1) V $\mathbb{Z}$-graded vector space

Let $C(V) = \bigoplus_{n\geq 1}(V[1])^{\otimes n}$ $\longrightarrow$ "formal manifold" $\longrightarrow$ "dual or predual to alg. of fns"

cofree coalgebra cogenerated by $V[1]$

coproduct $\Delta(v_1 \otimes \cdots \otimes v_n) = \sum_k (v_1 \otimes \cdots \otimes v_k) \otimes (v_{k+1} \otimes \cdots \otimes v_n)$

(2) square zero coderivation.

$$C(V) \xrightarrow{Q} C(V)[1] \longrightarrow [Q,Q]$$ "vector field"

Pf: Q determines (and is determined by) values on cogenerators.

$$\begin{array}{ccc} V[1]^{\otimes n} & \longrightarrow & V[1][1] \\ & \updownarrow & \\ V^{\otimes n} & \longrightarrow & V[2-n]. \end{array}$$

Working out the algebra $\rightsquigarrow$ T_{A}

(Paul: There's this Koszul duality of algebras...)

Def: A_∞ morphism

$$\begin{array}{ccc} C(V_1) & \xrightarrow{Q_1} & C(V_1)[1] \\ F \downarrow & & \downarrow F[1] \\ C(V_2) & \xrightarrow{Q_2} & C(V_2) \end{array}$$

equivalently:

$F : C(V) \longrightarrow C(W)$

$F_n : V^{\otimes n} \longrightarrow W[1-n]$ "Taylor coeffs of F" satisfying a bunch of equations involving

$m_n^{V_1}$'s, $m_n^{V_2}$'s, F_n's.

First 2 eqn's are:

- $d \circ F_1 = F_1 \circ d$.
- $F_1(v_1) \cdot F_1(v_2) = F_1(v_1 \cdot v_2) \pm F_2(d(v_1), v_2) \pm F_2(v_1, d(v_2))$.

"Perturbation lemma/transfer lemma":

Let (A, m_n) be an A_∞ algebra.

Let $\pi : A \to A$ morphism of complexes (not nec. an alg. morphism) satisfying

$$\pi^2 = \pi, \quad d\pi = \pi d.$$

Assume there is a chain htopy $H : A \to A[-1]$

$$1 - \pi = dH + Hd$$

Let B be the image of π.

Then the A_∞ structure on A induces one on B.

Example:
$$A^\bullet = (\ker d^\bullet) \oplus \mathrm{comp}_1^\bullet$$
$$= (\operatorname{im} d^{\bullet-1} \oplus \underbrace{\mathrm{comp}_2^\bullet}_{\cong H^\bullet(A,d)}) \oplus \mathrm{comp}_1^\bullet$$

(Paul: doesn't need to be done over a field, but then you need to be careful, e.g. $a \otimes a \cdots \otimes a$ is derived tensor prod. it works though!)

$$\Pi : A^\bullet \xrightarrow{p} H^\bullet(A,d) \xrightarrow{i} A$$

$H : (\mathrm{im}\, d^{\bullet -1}) \oplus \mathrm{comp}_2^\bullet \oplus \mathrm{comp}_1^\bullet \xrightarrow{\quad} d^{-1} \oplus 0 \oplus 0$
$(\mathrm{im}\, d^{\bullet -2}) \oplus \mathrm{comp}_2^{\bullet -1} \oplus \mathrm{comp}_1^{\bullet -1}$

$B = H^\bullet(A,d)$, so get A_∞ structure on B.

<u>Paul</u>: 2 ways to think about this

(1) Geometric way: analogue of normal form thm for vec. fields:
given vec. field, non-deg 0, can make coord. change so it's linear; at least partially linearize if only non-deg. on some directions.

(2) works only for cyclicly invariant guys)
A ~ degs. of freedom of field

this corresponds to cutting off massive modes to get an effective field theory.

Let $i: B \to A$ inclusion

Let $p: A \to B$ projection. Then:

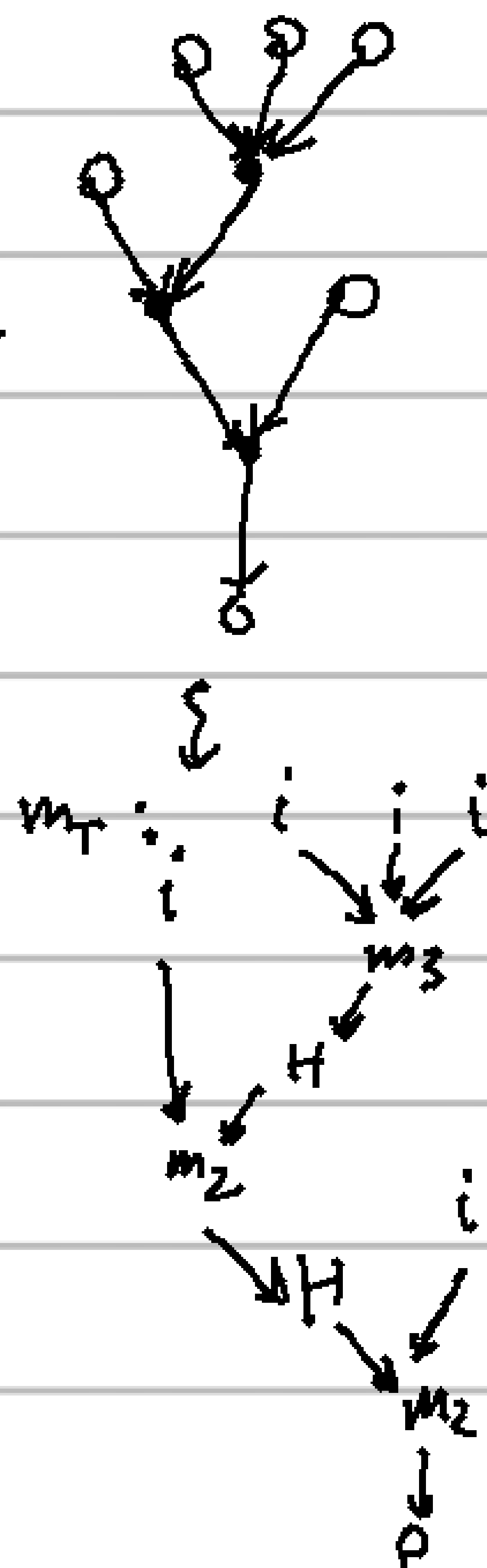
$$m_1^B = p \circ m_1 \circ i$$

$$m_2^B = p \circ m_2 \circ (i \otimes i)$$

$$m_n^B \rightsquigarrow \sum_{T} \pm m_T \qquad n \geq 3$$

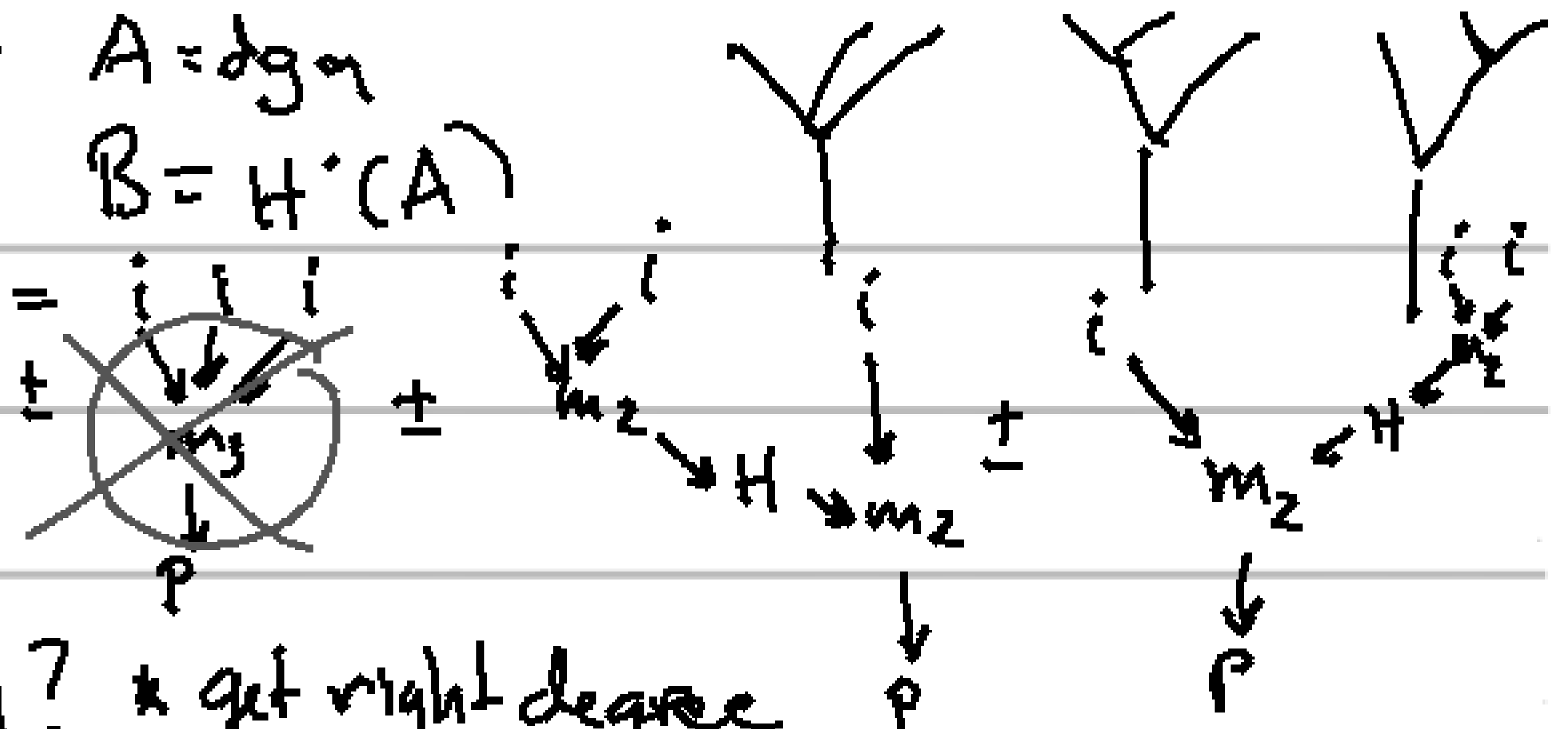
T oriented planar rooted tree w/ n tail vertices, such that all internal vertices have #incoming edges ≥ 2.

Ex:



example: $A = $ dga
$B = H^\bullet(A)$

$m_3^B =$



(why H? * get right degree
* otherwise A_∞ assoc. equations might imply everything vanishes)

triple Massey product

Rest of proof: gymnastics.

Prop: There are A_∞ quasi-isoms $F: A \to B$, $G: B \to A$ such that $F_1: A \to B$ is p, $G_1: B \to A$ is i.

Proof: Do some more gymnastics, sum over all possible trees.

Cor: A_∞-quasi-isos. are invertible.

Pf: $A \xrightarrow[A_\infty \text{ q.i.}]{} B$ $\qquad$ $\underset{d=0}{H^\bullet(A)} \xrightarrow[A_\infty\text{-iso.}]{} \underset{d=0}{H^\bullet(B)}$

so, map is honestly invertible.

i.e. if $C(V) \longrightarrow C(W)$ is an iso. on cogenerators, then it's an iso. in the A_∞ category.

Construct the inverse:

$$\begin{tikzcd} A \arrow[r] & B \arrow[d, "\wr"] \\ H(A) \arrow[u, "\wr"] & H(B) \arrow[l, "\sim"', "\text{inverse iso.}"] \end{tikzcd}$$

Example: Thm: X compact Kähler mfold, then

$$\begin{tikzcd} (\Omega^\bullet(X), d) \arrow[r, "\text{dga}"', "\text{q.i.}"'] \arrow[rr, bend right, "A_\infty \text{ quasi-iso.}"'] & \bigcirc & (H^\bullet(X), d=0) \arrow[l, "\text{dga}", "\text{q.i.}"'] \end{tikzcd}$$

(Deligne-Griffiths-Morgan-Sullivan)
not using A_∞ language.

$\Rightarrow$ (1) the A_∞ structure on $H^\bullet(X)$ is trivial.

e.g. Massey products are all zero.

(2) Rat. htpy type of a cpct. Kähler m'fold is a formal consequence of its cohomology ring.

(Paul: Hole in the literature. DGMS does this, but no one's written down the A_∞ version. Merkulov doesn't quite do it.)

(John: defense of dga's: Can consider a subcategory (fibrant/cofibrant objects in that category) where things actually do invert.

(Paul: All A_∞ algebras are fibrant, so there are some choices when representing htpy category of dga's.).

NB: middle guy (1) for Kähler quasi-iso. is $\ker d^*$.