Day 2 Talk 1: Joly, A_∞ structs. in topology

Very old + : intersection product.

$\cup : H^*M \otimes H^*M \longrightarrow H^*M$.

(graded) comm. assoc. alg.

w/ higher ops (Massey prods., Steenrod mod p)

$$\begin{tikzcd}
\cup : C^*M \otimes C^*M \arrow[dr] \arrow[rr] & & C^*M \\
& C^*(M\times M) \arrow[ur, "\Delta^*"'] &
\end{tikzcd}$$

dga (E_∞-alg)

E_∞ : commutative :: A_∞ : associative.

M smooth cpct. d-dim'l mfld

$\cap : H_{*+d}(M) \otimes H_{*+d}M \longrightarrow H_{*+d}M$.

"$\cap$": $C_{*+d}M \otimes C_{*+d}M \longrightarrow C_{*+d}M$"

Later: How to see A_∞ (& E_∞?) structure at the level of Morse theory.

M^{-TM} is E_∞ ring spectrum $H_*M^{-TM} \cong H_{*+d}M$

Old $(A_\infty^{\text{original}})_-$: Pontryagin Product

X pointed space

$$\Omega X \qquad (\gamma_1 \circ \gamma_2)(t) = \begin{cases} \gamma_1(2t) & t \le \frac{1}{2} \\ \gamma_2(2t-1) & t \ge \frac{1}{2} \end{cases}$$

based loops

<u>Observations</u>. (sets $r = 1$)

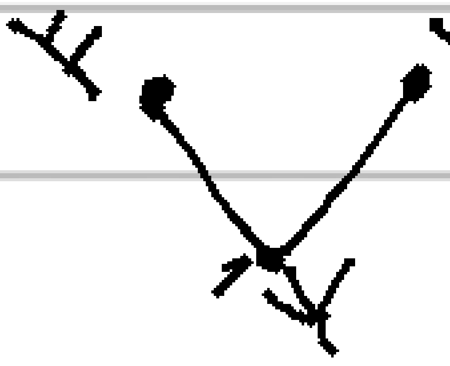
arc length, $\gamma : [0,r] \to X$.

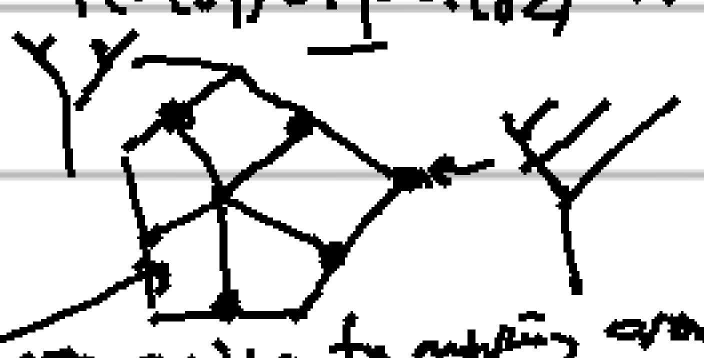
$$\Omega X \xrightarrow{i} \Omega^M X = \{(r,\gamma) \in \mathbb{R}_{\ge 0} \times \Omega X : \text{if } r = 0,\ \gamma \text{ is constant}\}$$
$$\Omega^M X \xrightarrow{p} \Omega X$$

$$\begin{tikzcd} \Omega^M X \arrow[r, bend left, "id"] \arrow[r, bend right, "i \circ p"'] \arrow[r, phantom, "\Uparrow H"] & \Omega^M X \end{tikzcd}$$

$$m_2(\gamma_1,\gamma_2) = p(i(\gamma_1) \circ i(\gamma_2))$$

$$m_2(m_2(\gamma_1,\gamma_2),\gamma_3) = p(\underline{i\ p}(i(\gamma_1) \circ i(\gamma_2)) \circ i\, \gamma_3)$$
$$\Downarrow$$
$$p(i(\gamma_1) \circ i(\gamma_2) \circ i(\gamma_3))$$
$$\Uparrow$$
$$m_2(\gamma_1, m_2(\gamma_2,\gamma_3)) = p(i(\gamma_1) \circ \underline{i\ p}(i(\gamma_2) \circ i(\gamma_3)))$$





cells corresponding to moving around $i \circ p$'s!

metrized trees w/ lengths in $[0,\infty]$ give multiplications on ΩX.

Interior of a cubical subdivision of K_n.

i.e. $p(H_t(i(\gamma_1)\circ i(\gamma_2))\circ i\,\gamma_3)$.

- Easy to ~~[illegible]~~ transfer strict structures

- Thm (~~[illegible]~~ principle):
  Recognition

Y has htpy type of a loop space

$\Updownarrow$

Y has the htpy type of a top. monoid w/ π_0 group.

$\Updownarrow$

Y has an A_∞-space structure w/ π_0 a group

Left: "B" for A_∞ & $\Omega B \underset{\substack{\uparrow \\ \text{gp.-like } A_\infty}}{Y} \simeq Y$

Newer: Loop product (string)

$$\begin{tikzcd}
\Omega M \arrow[r, "i"] & LM \arrow[d, "ev_0"] \\
& M \arrow[u, dashed, bend left, "const."]
\end{tikzcd}$$

fiber diagram.

Serre SS:

$$\begin{tikzcd}
 & H_{*+d}(M; H_*\Omega M) \arrow[d, Rightarrow] \\
A_\infty \text{ alg.} \arrow[ur] & H_{*+d}(LM) \\
 & A_\infty \text{ alg?} \arrow[u]
\end{tikzcd}$$

what about $C_{*+d}(LM)$?

would be nice to have

$$H_{*+d}M \to H_{*+d}LM \xrightarrow[\text{deg-d.}]{?} H_*$$

$$\begin{tikzcd}
\Omega M \arrow[r, "\text{codim d.}"] \arrow[d] & LM \arrow[d] \\
* \arrow[r, "\text{codim d}"', "\text{emb.}"'] & M.
\end{tikzcd}
\Rightarrow H_{*+d}LM. \to H_*\Omega M.$$

"intersect a cycle w/ ΩM"

$$\begin{tikzcd}
 & LM \\
LM \times_M LM \arrow[ur] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & LM \times LM \arrow[d] \\
M \arrow[r, "\Delta"'] & M^2
\end{tikzcd}$$

$$\begin{tikzcd}
H_{*+d} LM \otimes H_{*+d} LM \arrow[d, "\times"] \\
H_{*+2d}(LM \times LM) \arrow[d, "\text{intersect w/ ``}LM \times_M LM\text{''}"] \\
H_{*+d}(LM \times_M LM) \arrow[d, "\text{compose loops}"] \\
H_{*+d}(LM).
\end{tikzcd}$$

Claim: $HH(C^* LM) = C^* \Omega M$??

- How do we see $\cup$?

Geometric idea:

$[p] \in C_i(M, f) \xleftrightarrow{\text{think}} [W^s_f(p)]$ ↑ i dim'l.

$[p] \in C^i(M, f) \longleftrightarrow PD([W^u_f(p)])$ ↖ d−i dim'l.

$$\underset{\substack{\uparrow \\ C^{m(p)}}}{[p]} \cup \underset{\substack{\uparrow \\ C^{m(q)}}}{[q]} = \sum_{\substack{[r] \text{ crit. pt.} \\ m(r) = m(p) + m(q)}} ([p] \cup [q])(\underset{\substack{\uparrow \\ C_{m(r)}}}{[r]}) \cdot [r] \in C^{m(r)}$$

$$= \sum_r \#(W^u(p) \cap W^u(q) \cap W^s(r)) \cdot [r]$$

↑ BAD: disjoint or highly non-transverse

Solution: f, g, h "generic" Morse fcn.

$p \in \mathrm{Crit}\, f$

$q \in \mathrm{Crit}\, g$

$r \in \mathrm{Crit}\, h$

$$\sum_r \#\left(W^u(p) \cap W^u(q) \cap W^s(r)\right) \mathbb{Z}$$

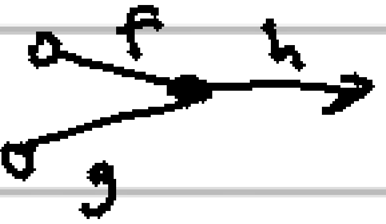
~~again~~

G7

Associative?

f_1, f_2, f_3, f_4

Think of as

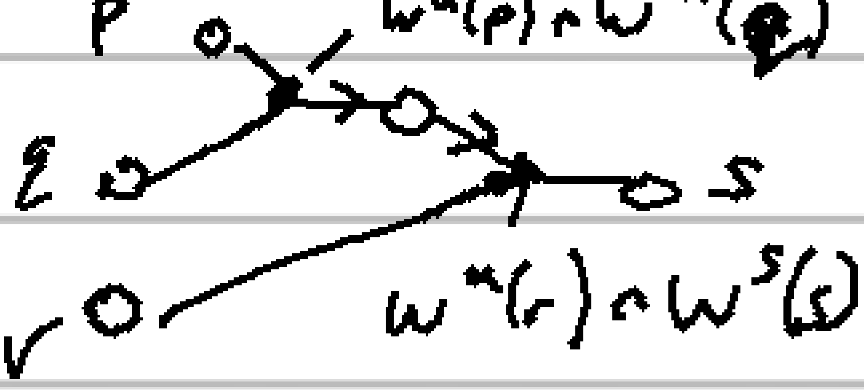
Hom ( , M)

ev. at middle pt.

"Graph Flows"

M

Higher composition:



$$\begin{tikzcd}
F \arrow[d, hook] & & \\
\mathbb{R}^{\ell}_{>0} \times \left(W^u(p) \cap W^u(q)\right) \arrow[rr] \arrow[d] & & M \\
F \arrow[rr] & & W^u(r) \cap W^s(s) \arrow[u, hook]
\end{tikzcd}$$

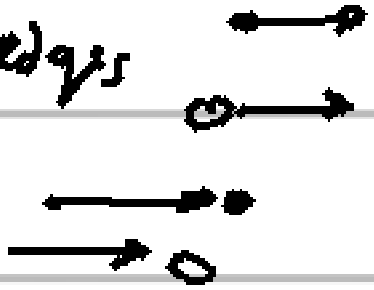
Hom ( , M)

$\mathcal{M}_{p,q}$:= moduli spaces of "metric graphs"
w/ "missing vertices"

$\downarrow$ cpct, orients

$\downarrow$ length on cpct edges

+ smooth function on each edge.
w/ p numbered univalent edges
q outgoing edges



slogan:

$C \in \mathcal{M}_{p,q} \rightsquigarrow$ family of cobordisms from p pts. to q pts.

$\uparrow$ cycle.

$$C_*(X)^{\otimes p} \longrightarrow C_*(X)^{\otimes q}.$$

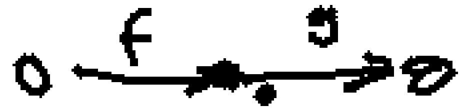
1) closed endpoints, then you can get these maps either H_* or on generic chains

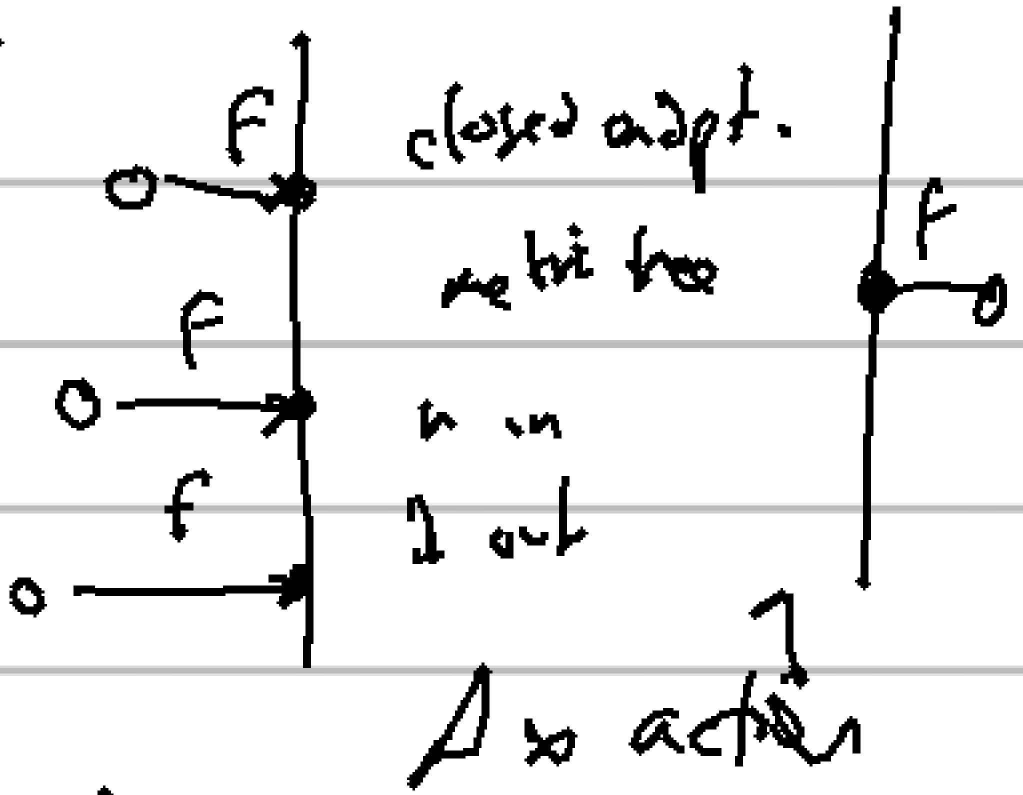
2) Open endpoints: get

$$\bigotimes_{i \text{ incoming}} C_*(X, f_i) \longrightarrow \bigotimes_{i \text{ out}} C_*(X, f_i)$$

Change of phase tr.

(fig) generic.





To get bigger things (Eno action, steerod operating), more freedom.