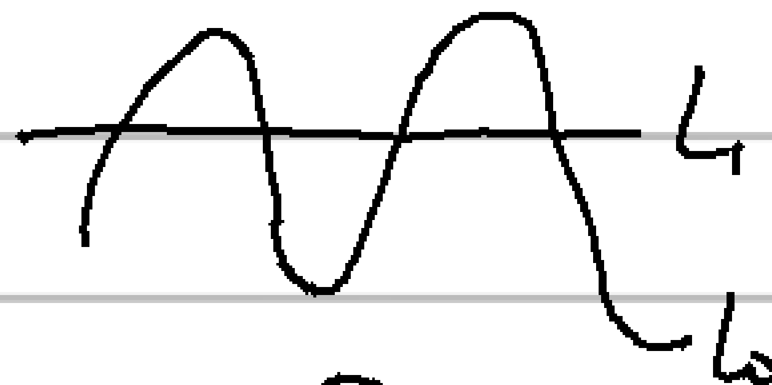
Day 2 Talk 2: Hiro

<u>Setting</u>: (M, ω) $\begin{cases} \dim M = 2n, \\ \omega \text{ sympl. form} \end{cases}$ $\begin{pmatrix} 2 \text{ form,} \\ d\omega = 0 \\ \text{non-deg.} \end{pmatrix}$

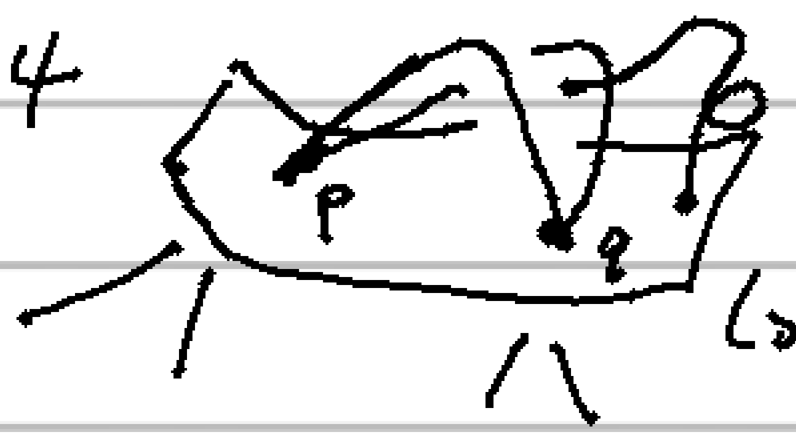
$L_0, L_1 \subset M$ Lagrangian submanifolds
$(\omega|_{L_0}, \omega|_{L_1} = 0)$

$L_0 \pitchfork L_1$

<u>Pictures</u>: (1) $M = \mathbb{R}^2$



(2) $\dim M = 4$



<u>Morse Theory</u>

(1) The Manifold

$$\mathcal{L}M = \{x \in C^\infty([0,1], M) \text{ s.t. } x(0) \in L_0,\ x(1) \in L_1\}$$

Choose $\underline{p} \in \mathcal{L}M$, constant path @ $p \in L_0, L_1$

$\tilde{x} \in \widetilde{\mathcal{L}M}$ can be written as $(x, [u])$ where $x \in \mathcal{L}M$, $\pi \tilde{x} = x$, u a homotopy from $\underline{p}$ to x.

② The function

$$u : [0,1]\times[0,1] \to M$$

$$A(x,[u]) = \int_0^1\int_0^1 u^*\omega$$

Exercise: A is well-defined

③ The metric: Let J be an a.c.s.

($\omega(-, J-)$ is a Riem. metric)

$$T_x \mathcal{L}M = \{\text{vector fields } X \text{ along } x \text{ s.t. } X(x_0)\in T L_0,\ X(x_1)\in T L_1\}$$

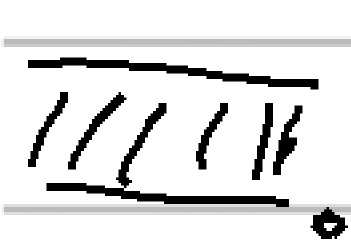
So, $$\tilde{g}(X, Y) = \int_0^1 \omega(X(t), JY(t))\,dt.$$

Exercise: ① The critical points of this A are given by constant paths into $L_0\cap L_1$

② The gradient flow is given by maps

$u : [0,1]\times \mathbb{R} \to M$,

J-hol. maps such that



$u(1,t)\in L_1$, $\lim_{t\to\infty} u(s,t) = q$

$u(0,t)\in L_0$, $\lim_{t\to-\infty} u(s,t) = p$

Given two almost $\mathbb{C}$-manifolds M_1, M_2
$J_1 \quad J_2$,

$u : M_1 \to M_2$ is J-hol if

$du \circ J_1 = J_2 \circ du$.

To define differential, we'd like to know

$$\dim \underset{\{\text{fns from } p \text{ to } q\}}{\mathcal{M}(p,q)}$$

Then we can define the differential,

$$\delta p = \sum_{\dim(\hat{\mathcal{M}}(p,q))=0} n_{pq} \cdot q$$

$n_{p,q} = \# \hat{\mathcal{M}}_{p,q}$

$\hat{\mathcal{M}}(p,q) = \mathcal{M}(p,q) / \text{parametrizations}$

Paul:

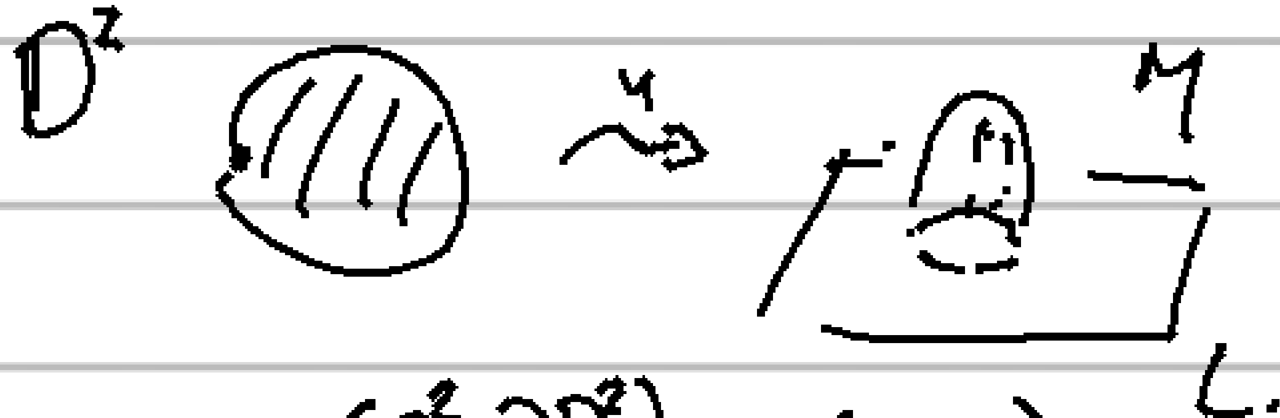
Due to polarized manifolds, it's in general impossible to work w/ $\mathbb{Z}$ coeffs, & when it's possible, requires some choices.

Today, we work w/ $\mathbb{Z}/2$ coeffs.

Given some $u \in \mathcal{M}(p,q)$, how do we find the dimension of $\mathcal{M}(p,q)$ on this component?

Answer: Maslov index

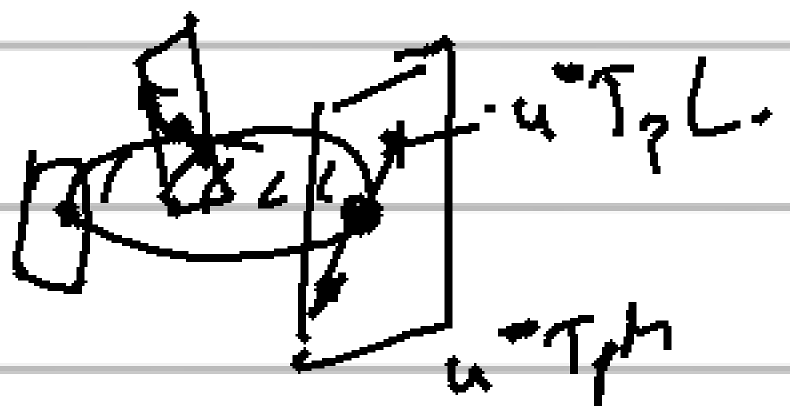
Small project: Learn about Maslov index.



$u : (\mathbb{D}^2, \partial \mathbb{D}^2) \to (M, L)$.

We can pull back TM onto $\mathbb{D}^2$

TL onto $\partial \mathbb{D}^2$.

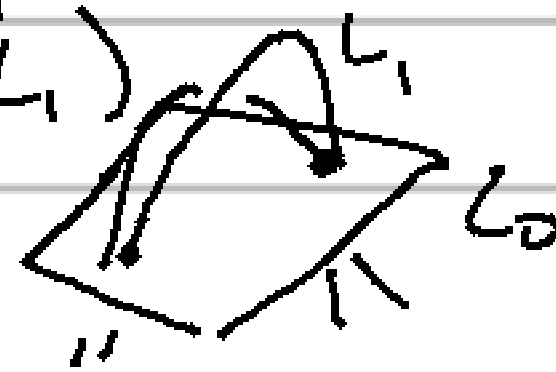


walking along the boundary gives us a map

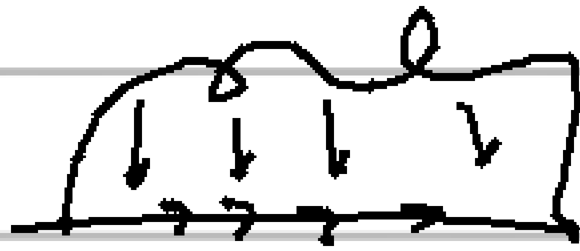
$S^1 \longrightarrow Gr^{Lag}$

knowing $\pi_1(Gr^{Lag}) = \mathbb{Z}$ we can compute degree of this map.

How do define this for (M, L_0, L_1)

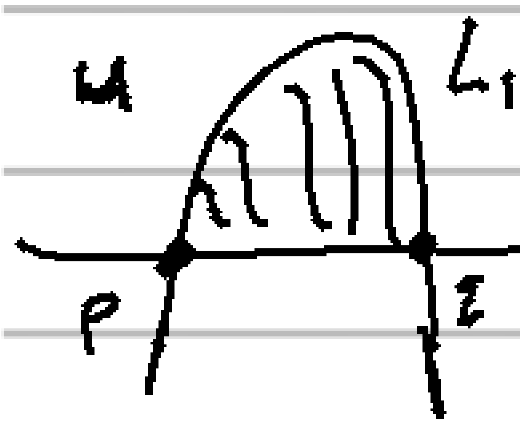


Trivializing u^*TM gives $D^2 \times \mathbb{C}^n$.
choose a picture like this.



Choose trivialization of (TL_0, TM) along bottom path

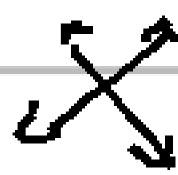
In $\mathbb{R}^2$ (Example)



before reparametrization, the dim is dim 1.

L_0 Maslov index$(u_{p\to q}) = 1$.

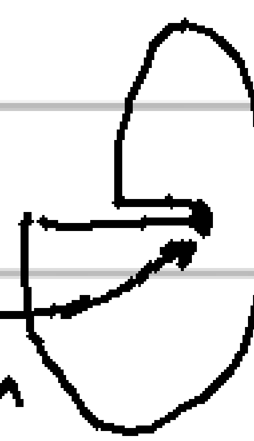
(dim. of $\mathcal{M}(p,q)$ before reparam.)



what's dim $\mathcal{M}(p,r)$?

ex. 2.
dim = 2.

have strip

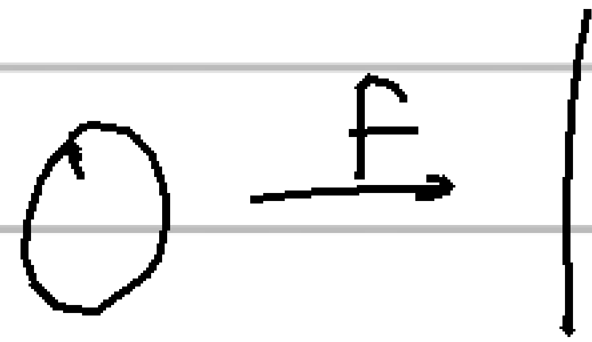


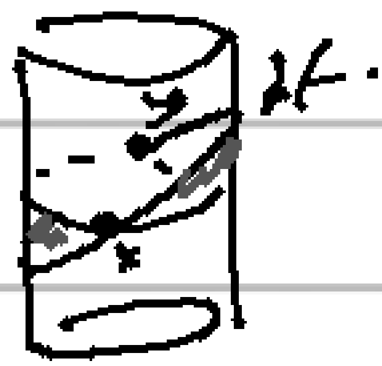
1 degree of freedom

Take $M = T^*S^1$

L_0 = zero section

$L_1 = \text{graph}(\partial f)$





Looking at a wine bottle, we see

$\check{M}(x,y) = 2$ points,

$\partial x = \partial y = 0.$

$$\begin{array}{ll} & 0 \\ & \searrow \\ x & \mathbb{Z}/2\mathbb{Z}. \\ & \downarrow 0 \\ y & \mathbb{Z}/2\mathbb{Z} \\ & \downarrow \\ & 0 \end{array}$$

$\partial y = 0$

$HF^1(L_0, L_1) = \mathbb{Z}/2\mathbb{Z}$

$HF^0(L_0, L_1) = \mathbb{Z}/2\mathbb{Z}$

<u>Thm</u> (Floer): Let M be a smooth mfld. For a "small fn," $HF^*(M, df) \cong H^*(M; \mathbb{Z}/2)$. zero section in T^*M.

($\mathbb{Z}/2\mathbb{Z}$ coeffs.)

f "small" means $\exists\ \varepsilon > 0$ s.t. $\forall x \in M$,

$|f|, |\nabla f|, |\nabla\nabla f| < \varepsilon$.

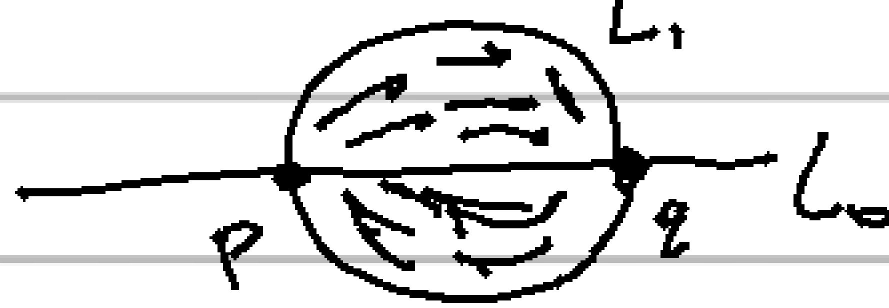
Why?

J-hol. discs correspond to flowlines

If $\pi_2(M, L) = 0$, M = zero section

Corollary: $HF^*(L, L) = H^*L$ ..

Non-example:



$\partial p = q, \ \partial q = p, \ \partial^2 \neq 0$.