Product Structures on Floer cohomology:

Fix (M,ω). For $L_0, L_1 \subset M$, we have

$$HF^*(L_0,L_1) = H^*(CF^*(L_0,L_1),\ \partial = \mu^1).$$

Def'n: Assume $L_0 \pitchfork L_1$. Then,

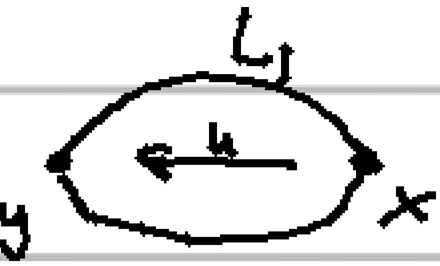
formal generator

$$CF^*(L_0,L_1) = \bigoplus_{x\in L_0\cap L_1} \Lambda\cdot x$$

$$\Lambda = \left\{ \sum_{k=0}^{\infty} a_k t^{\nu_k} \;\middle|\; a_k \in \mathbb{Z}/2,\ \nu_k \in \mathbb{R},\ \nu_k \to \infty \right\}.$$

(Doing this with coverings is painful; need choice of basept & one & these get in your way).

$$\partial(x) = \sum_y n(x,y)\, y;\quad n(x,y) = \sum_{\nu\in\mathbb{R}} t^{\nu} n_\nu(x,y).$$

n_ν = # of isolated 

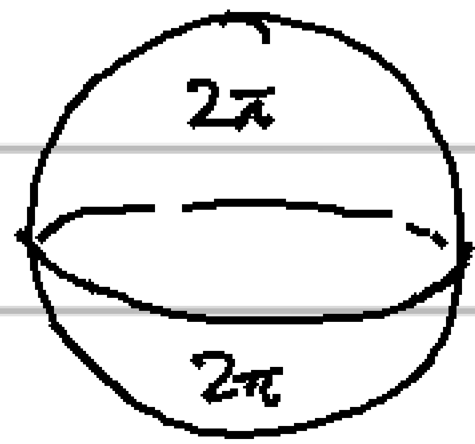
, with

$$E(u) = \int_{\mathbb{R}\times[0,1]} u^*\omega = \nu.$$

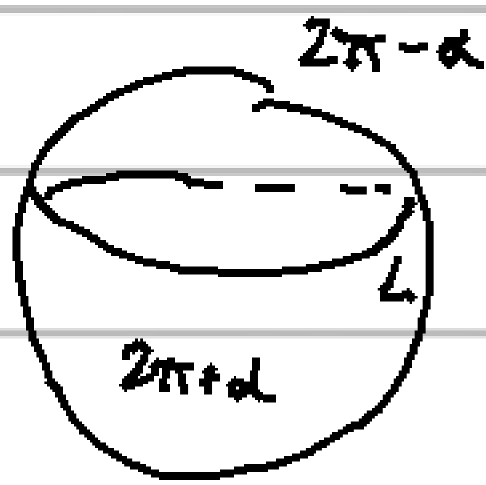
Main property: invariant under exact Lagrangian (Hamiltonian) isotopies of either L_0 or L_1.

(NB: Non-exact isotopies change area of disks, so maybe prevent things from canceling or vice versa).

Ex:



$L = L_0 = L_1$ $\quad HF^*(L, L) = H^*(S^1; \Lambda)$.



$\alpha > 0 \qquad HF^*(L, L) = 0$.

Some notes: Need to perturb L_i to be transverse, can always do this w/ Ham. isotopies.

Also, differential always has $t^{>0}$, b/c

$$E(u) = \int_{\mathbb{R}\times[0,1]} u^*\omega = \int_{\mathbb{R}\times[0,1]} \omega\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial s}\right) ds\, dt$$

$$= \int_{\mathbb{R}\times[0,1]} \left|\frac{\partial v}{\partial t}\right|^2 ds\, dt > 0$$

So can you restrict to $\Lambda' = \{t^{>0} \text{ powers}\}$?
Yes, but not an isotopy invariant.

(as the maps giving quasi-isos from Hamiltonian isotopies has $t^{<0}$ terms).

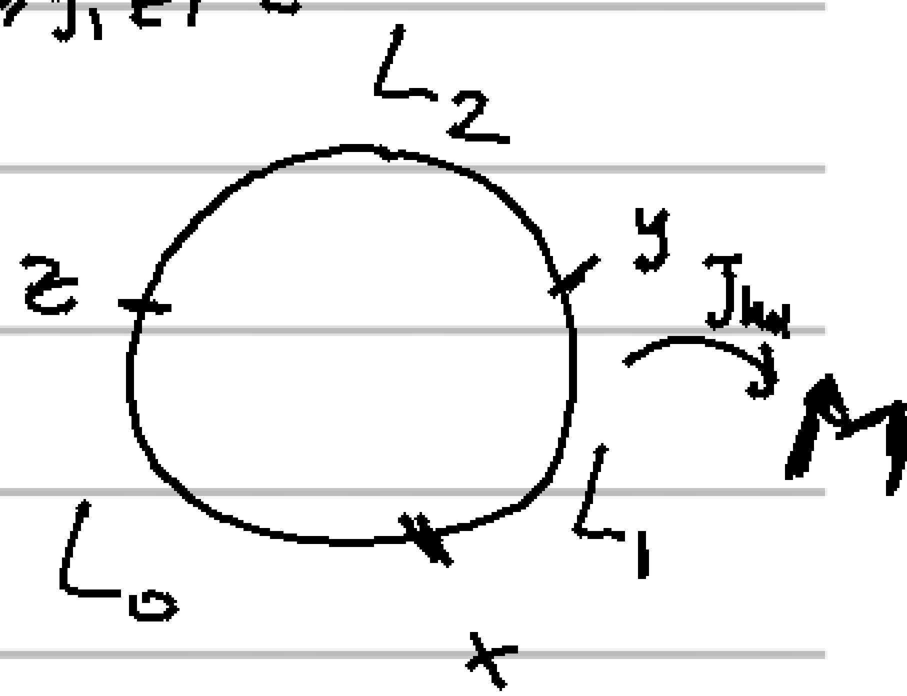
Product structures.

Take L_0, L_1, L_2 in general position.

$$CF^*(L_1,L_2)\otimes CF^*(L_0,L_1)\xrightarrow{\mu^2} CF^*(L_0,L_2)$$

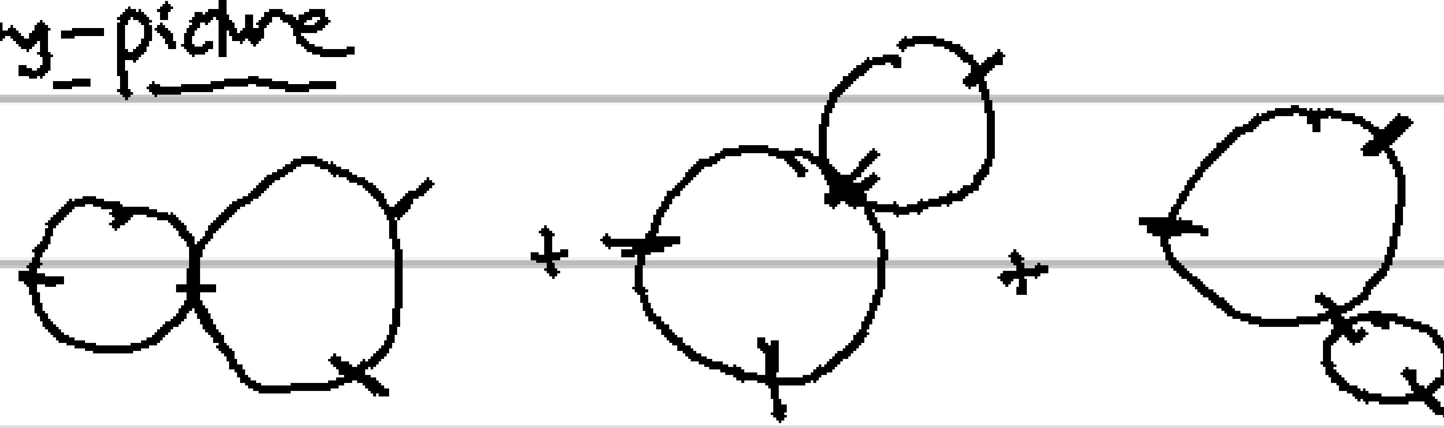
$$(x,y)\longmapsto \sum_z n(x,y,z)\, z$$

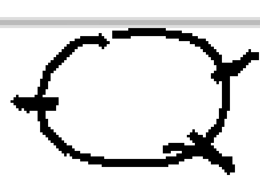
$n(x,y,z)$ counts maps (taking into account energies)



Lemma: μ^2 is a chain map.

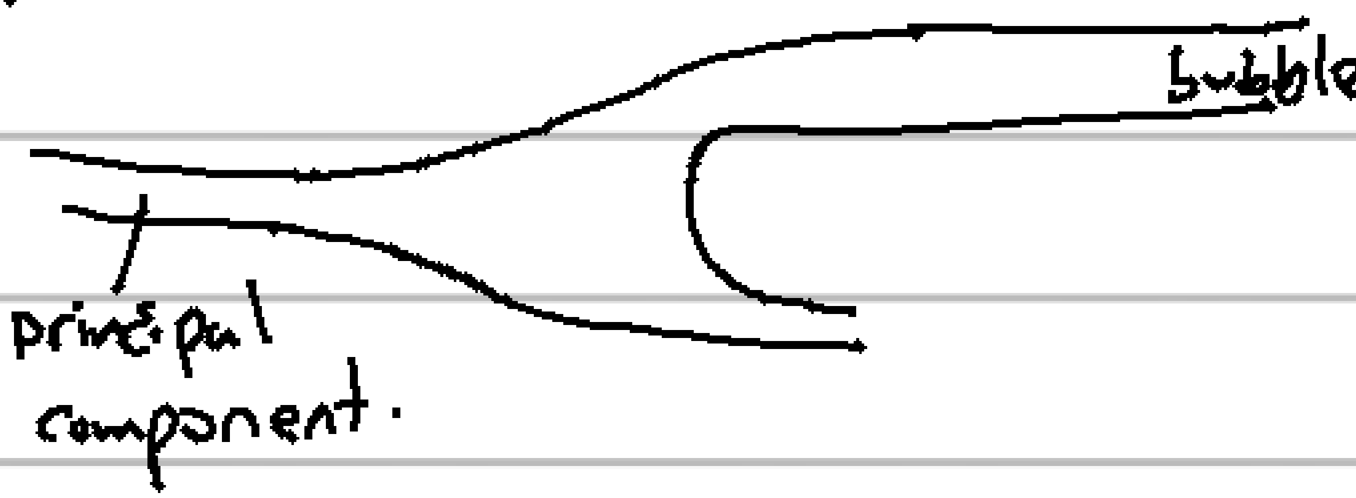
Proof-by-picture



$= \partial$  $= 0.$

i.e. $\mu^1(\mu^2) + \mu^2(\mathbb{1} \otimes \mu^1 + \mu^1 \otimes \mathbb{1}) = 0.$

limit process above has:



Define the Donaldson-Fukaya category

objects: $L \subset M$

morphisms: $\mathrm{Hom}(L_0, L_1) = HF^*(L_0, L_1)$

(if necessary perturbing to make them transverse),

composition induced by μ^2.

<u>Fact</u>: This category is unital (even though you're perturbing, $\exists$ identity morphisms on homology level).

<u>Fact</u>: $\dfrac{\text{Symplectic aut.}}{\text{Ham. isotopy}}$ acts on this category ("weak action")

<u>NB</u>: If you up this ↑ to a 2-group, acts nicely.

& thm's are well-defined up to canonical isom.

Introduced by Donaldson following ideas of Segal, but can't do anything with it.

- Gives a disconnected picture of structure ... can't really say much...

Pass to the chain level

Simplified partial version:

Fix a finite ordered collection $(L_1, \ldots, L_m)$ of Lagrangian submanifolds. Define the Directed Fukaya category $\mathcal{F}^{\rightarrow}(L_1, \ldots, L_m)$ as follows:

objects $\{L_1, \ldots, L_m\}$

morphisms $\hom(L_i, L_j) = \begin{cases} CF^*(L_i, L_j), & i<j \\ \Lambda \cdot e_{L_i}, & i=j \\ 0 & i>j \end{cases}$

($i=j$ part is formal. this avoids dealing with $\mathrm{Hom}(L_i, L_i)$; need to throw away $\hom(L_j, L_i)$ $j>i$ to ensure we can't compose back to $\mathrm{Hom}(L_i, L_i)$).

This has the structure of an A_∞ category.

$\mu^1 : \hom(L_i, L_j) \circlearrowright$ Floer differential ($i<j$) or zero

$\mu^2 : \hom(L_j, L_k) \otimes \hom(L_i, L_j) \to \hom(L_i, L_k)$

the holomorphic triangle product

(or trivial extensions which make e_{L_i} into a unit).

$\vdots$

$\mu^d : \hom(L_{i_{d-1}}, L_{i_d}) \otimes \cdots \otimes \hom(L_{i_0}, L_{i_1}) \to \hom(L_{i_0}, L_{i_d})$

nonzero only if $i_0 < \cdots < i_d$.

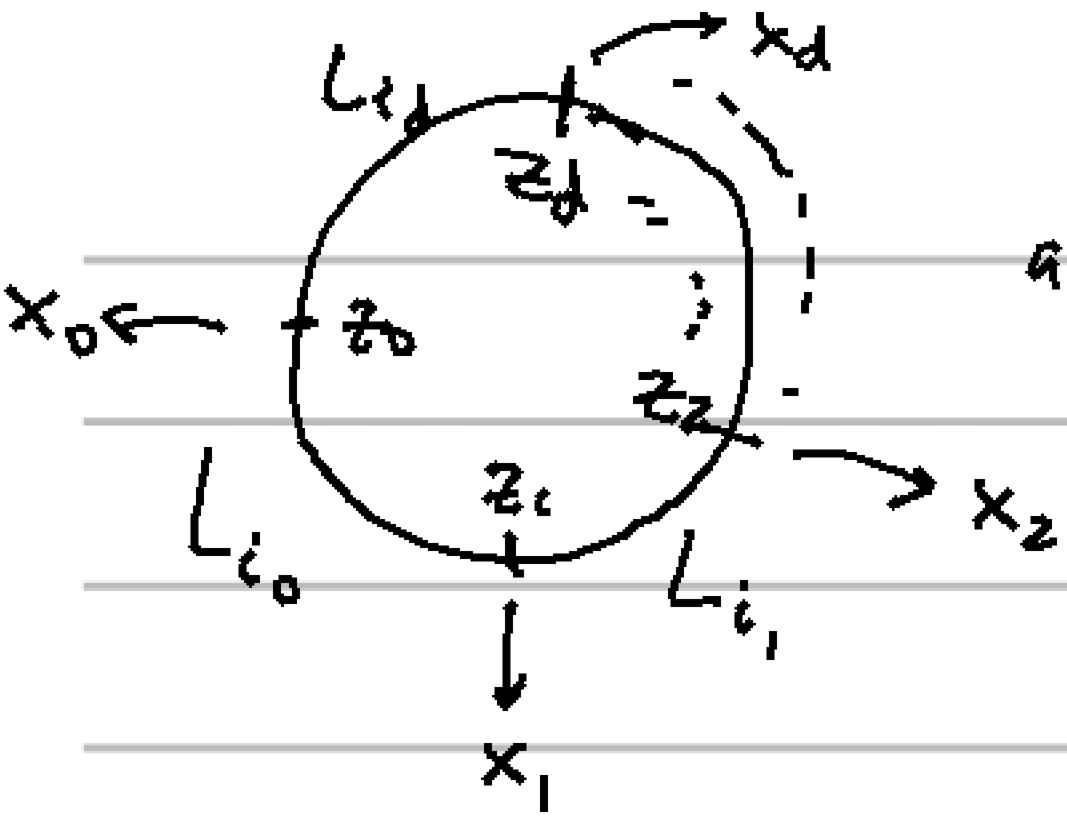
<u>Definition of μ^d</u>: Given $x_1 \in L_{i_0} \cap L_{i_1}$, $x_2 \in L_{i_1} \cap L_{i_2}$, ... (inputs) and $x_0 \in L_{i_0} \cap L_{i_d}$, and $\nu > 0$, consider:

$$\mathcal{R}^{d+1}(x_0, \dots, x_d)^{\nu} = \{ (S, z_0, \dots, z_d, u) \mid$$

S Riem. surface isomorphic to a closed disc

$z_0, \dots, z_d \in \partial S$ distinct cyclically ordered boundary pts

$u : S \setminus \{z_0, \dots, z_d\} \to M$ J-hol. maps with the following boundary and asymptotic conditions:



and $\int u^*\omega = \nu \Big\} \Big/ _{\text{iso.}}$

This comes with a forgetful map

$$\mathcal{R}^{d+1}(x_0,\dots,x_d)^\nu \longrightarrow \mathcal{R}^{d+1}$$

(NB: Need to let z_i vary above, e.g. to get A_∞ structure).

$$n(x_0,\dots,x_d)^\nu := \#\mathcal{R}^{d+1}(x_0,\dots,x_d) \in \mathbb{Z}/2$$

count isolated points, throw rest away

(NB: there are structures one can imagine where we don't throw the rest away, but for now we will).

$$n(x_0,\dots,x_d) \stackrel{\text{def}}{=\!=} \sum_\nu n(x_0,\dots,x_d)^\nu t^\nu \in \Lambda.$$

$$\mu^d(x_1,\dots,x_d) \stackrel{\text{def}}{=\!=} \sum_{x_0} n(x_0,\dots,x_d)\, x_0$$

these numbers $n(x_0, \dots, x_d)$ depend on J & possibly auxiliary choices

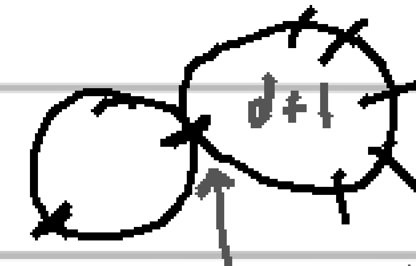
Thm: $\mathcal{F}^{\to}(L_1, \dots, L_k)$ is an A_∞ category.

Proof by picture: Uses compactifications

$$\overline{R}^{d+1}(x_0, \dots, x_d)^1 \longrightarrow \overline{R}^{d+1}$$

Generically, a path here hits faces of not corners.

$0 = \partial\Big($  $\Big) = \sum_{\substack{e+f=d+1 \\ e,f \geq 2,\ i}}$ 

$+ \sum_i$ 

Note: Don't need to treat these two separately, the differential is just μ^1, part of a sequence
(contrast to looking at A_∞ operad, where need to put in $\partial = \mu^1$ by hand, rest of μ^i are operad structure).

Getting an actual A_∞ category refining the Donaldson–Fukaya category.

(perturbations don't help!)

this problem already occurs in Morse theory!

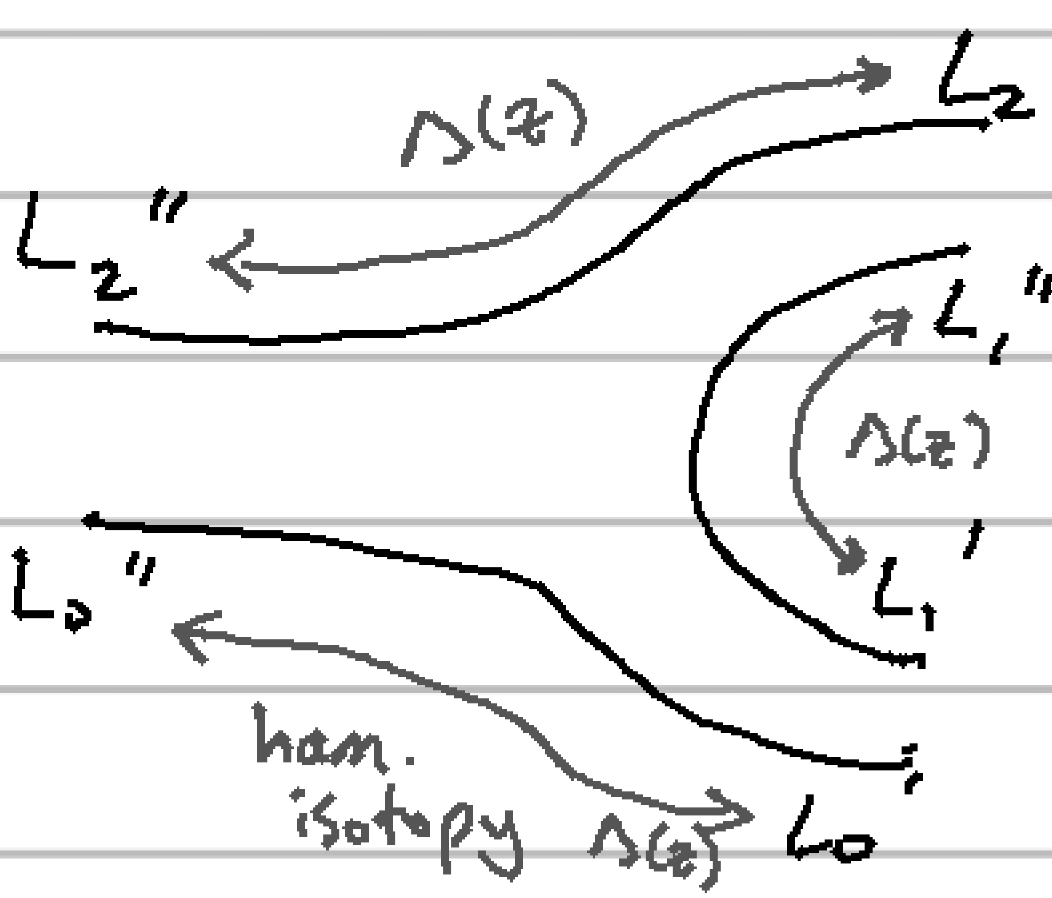
Involves choices of perturbations

For any (L_0, L_1), choose (L_0', L_1') which are Ham. isotopic & transverse.

$$\hom(L_0, L_1) = CF^*(L_0', L_1').$$

Problem:

$$\mu^2 : CF^*(L_1'', L_2'') \otimes CF^*(L_0', L_1') \longrightarrow CF^*(L_0'', L_1'')$$



$\Lambda : \partial D \setminus \{z_0, z_1, z_2\} \to \mathrm{Lag}(M)$ varies by Ham. isotopy

Consider maps $u : D^2 \setminus \{z_0, z_1, z_2\} \longrightarrow M$,

$u(z) \in \Lambda_z$, $z \in \partial D^2$ (moving boundary conditions)

u J-holomorphic

(obv. non-canonical)

Extend this to all Riem. surfaces that occur in a way that's consistent with compactification.

Thm: The resulting A_∞ structure is independent of all choices up to quasi-isomorphism.

When does this work as described?

- $[\omega]=0$, M noncompact but nice at ∞. ← $\omega = d\theta$, L exact
- $[\omega]=\lambda c_1$, $(\lambda>0)$ L "monotone" or "Bohr-Sommerfeld"
- $[\omega]=\lambda c_1$ $(\lambda<0)$, $2c_1$ divisible by $n-1$, $n=\dim_{\mathbb{C}}(M)$ ensures this is $A=\bigcirc$, $n+2c_1(A)-3\le 2$
- $c_1=0$, $n=\dim_{\mathbb{C}}(M)\le 2$. L's with vanishing Maslov class (borderline) min. alg. surfaces of general type

In each case, only a particular class of L's is allowed. a big L codim. condition.

Last case: Note: The reason why it fails is always the same! Each of these cases has different short term fixes.

no discs for generic J, but codim 1 phenomenon. Have to choose a point outside the walls to specify object in $\mathcal{F}(M)$.