Day 2 Talk 4: Nick S, Combinatorial Fukaya Categories

Setup: $(\underbrace{M, J}_{\text{Riemann surface (with boundary)}}, \underbrace{\omega}_{\text{any area form}}, \theta)$, $\omega = d\theta$.

Def'n: Let $(L_1, \ldots, L_n)$ be a collection of exact Lagrangian S^1's embedded in M.

$(\theta|_{L_i} = df_i)$

(don't have to worry about bubbling, can set $t \mapsto 1$ in Λ).

Exactness $\Rightarrow L_i$ not nullhomologous.

$\mathcal{F}(L_1, \ldots, L_n)$ over field $\mathbb{K}$, char $\mathbb{K} = 2$.

(let's not worry about gradings right now).

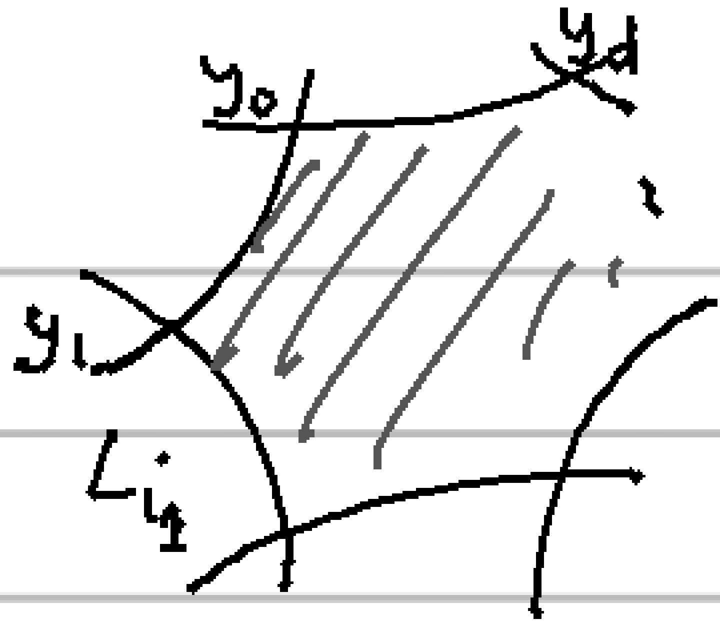
$$Ob(\mathcal{F}) = \{L_i\}$$

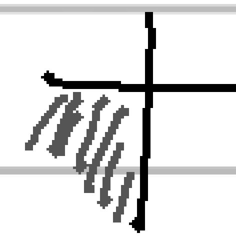
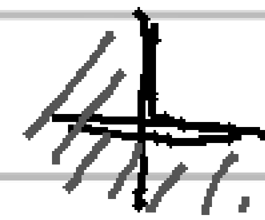
$$hom(L_i, L_j) = \begin{cases} \mathbb{K}^{L_i \cap L_j} & i < j \\ \mathbb{K} \cdot e & i = j \\ 0 & i > j \end{cases}$$

Define μ^d: Given $y_0 \in L_{i_0} \cap L_{i_d}$

$y_k \in L_{i_{k-1}} \cap L_{i_k}$ $(k = 1, \ldots, d)$

Let $n_d(y_0, \ldots, y_d) = \#$ immersed d-gons in M like this:



we want convex corners  , not 

Example of immersed 2-gon: 

$$\mu^d : \hom(L_{i_{d-1}}, L_{i_d}) \otimes \cdots \otimes \hom(L_{i_0}, L_{i_1}) \to \hom(L_{i_0}, L_{i_d})$$

$$\mu^d(y_d \otimes \cdots \otimes y_1) = \sum_{y_0} n_d(y_0, \to y_d)\, y_0$$

$(= 0$ unless $i_0 < \cdots < i_d)$

(why is this the same as $\mathcal{F}^{\to}(L_1, \dots, L_n)$?
Ans: essentially Riemann mapping thm.)

This is an A_∞ category: (using coderivation picture).

non-commutative non-unital co-algebra

$$T_{(L_1, \dots, L_n)} = \bigoplus_{d \geq 1} T_d$$

$$T_d = \bigoplus_{i_0 < \cdots < i_d} \hom(L_{i_0}, L_{i_1})^* \otimes \cdots \otimes \hom(L_{i_{d-1}}, L_{i_d})^*$$

(Define $\alpha \otimes \beta = 0$ if wrong ordering).

T_d has a basis

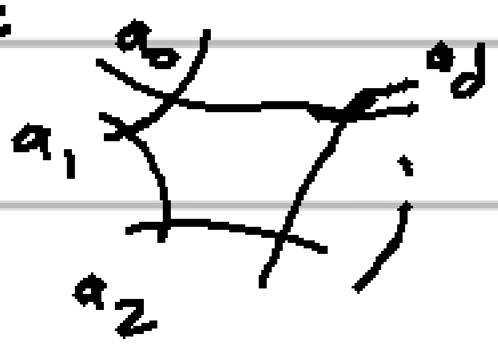
$$\{ a_1 \otimes \cdots \otimes a_d \mid a_j \in L_{i_j} \cap L_{i_{j+1}} \}$$
$$i_1 < \cdots < i_d$$

(really this is a dual basis).

Define $\delta : T \to T$ by

$$\delta(a_0) = \sum_d \mu_d(a_0, \ldots, a_d)\, a_1 \cdots a_d$$
$$(= \mu_1^* + \mu_2^* + \cdots).$$

For each polygon like this:



get a contribution to $\delta(a_0)$.

Extend $\delta|_{T_1}$ to T by Leibniz rule.

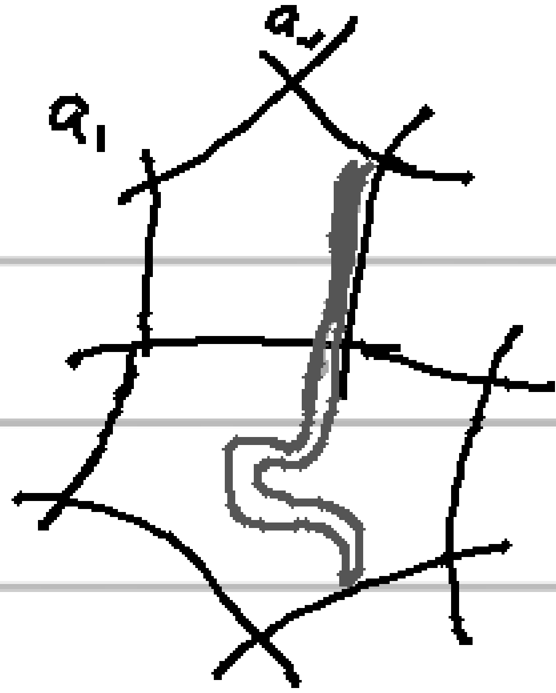
Prop: $\delta^2 = 0$

$$\delta^2 a_0 = \delta \sum_{\text{polys}} a_1 \cdots a_d$$

$$= \sum_i \sum_{\text{polys}} a_1 \cdots \delta a_i \cdots a_d$$

$$= \sum_{\text{polys}} \sum_{\text{polys}} a_1 \cdots a_{i-1} b_1 \cdots b_\ell a_{i+1} \cdots a_d$$

i.e. sum of things that look like



sum of polygons w/ one convex corner, i.e. 1 dim'l family, so take boundary & follow out.

⇒ terms in $\delta^2 a_0$ cancel in pairs

⇒ $\delta^2 a_0 = 0$ ⇒ $\delta^2 = 0$ (two things that go into the concave vertex must come out).

<u>Cor:</u> μ^d define A_∞ structure.
(exact Lag'ns key to ensuring a finite sum).

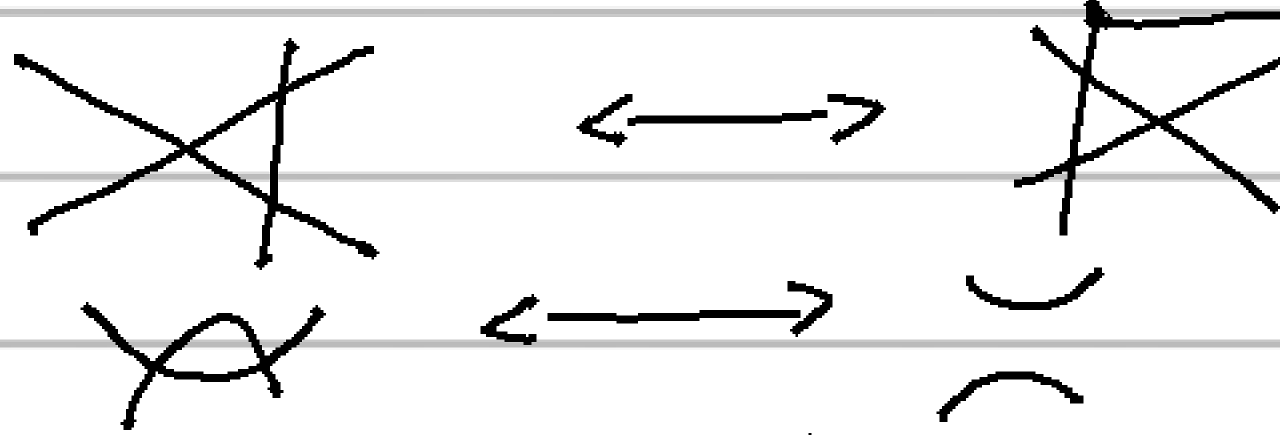
Prop: Hamiltonian isotopy of L_i changes $\mathcal{F}(L_1, \ldots, L_n)^{\rightarrow}$ by a quasi-isomorphism.

Pf:

Moves we worry about

– I.

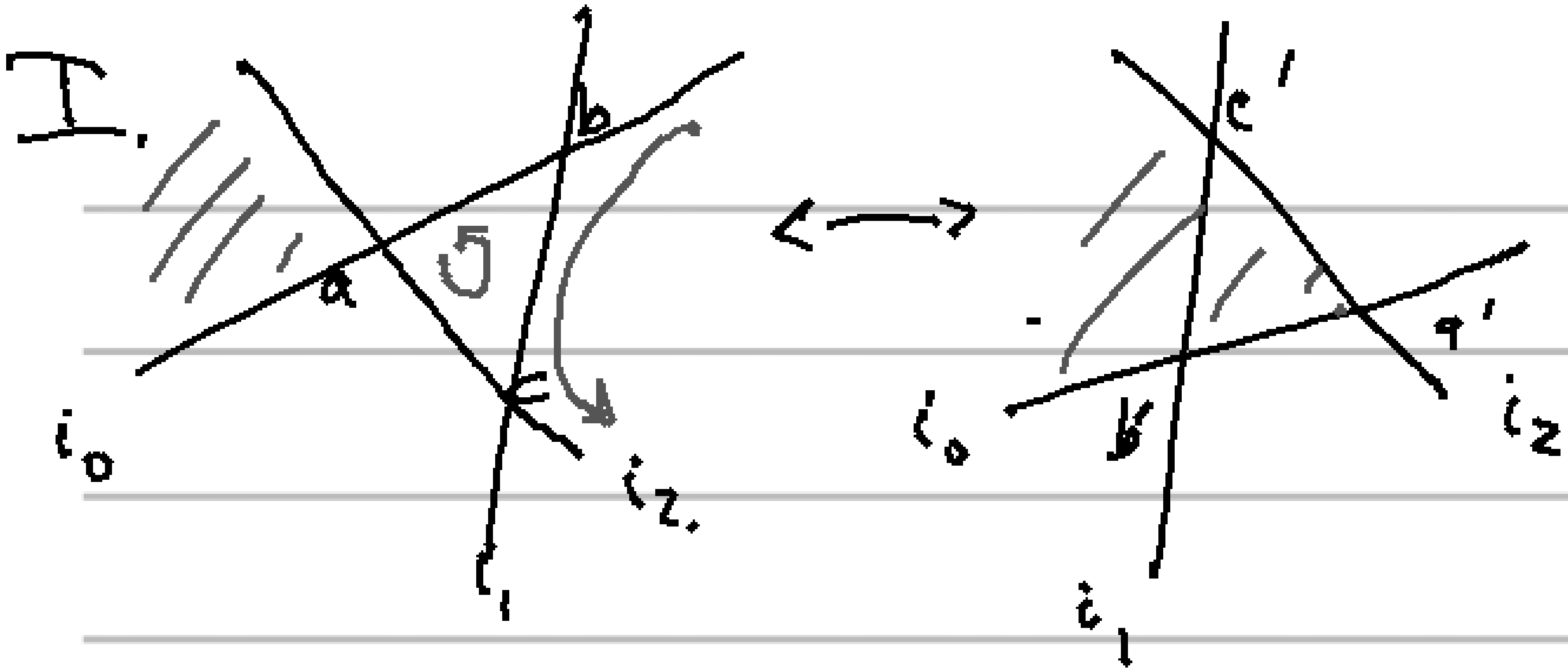
II.



(i.e. Reidemeister moves 2 & 3 but don't have I b/c Lag'ns are embedded).

Let's look at one of these cases.

I.



WLOG i_0 is smallest,

if not ($i_0 < i_1 < i_2$),

$a \mapsto a'$

$b \mapsto b'$

$c \mapsto c'$

if $i_0 < i_1 < i_2$:

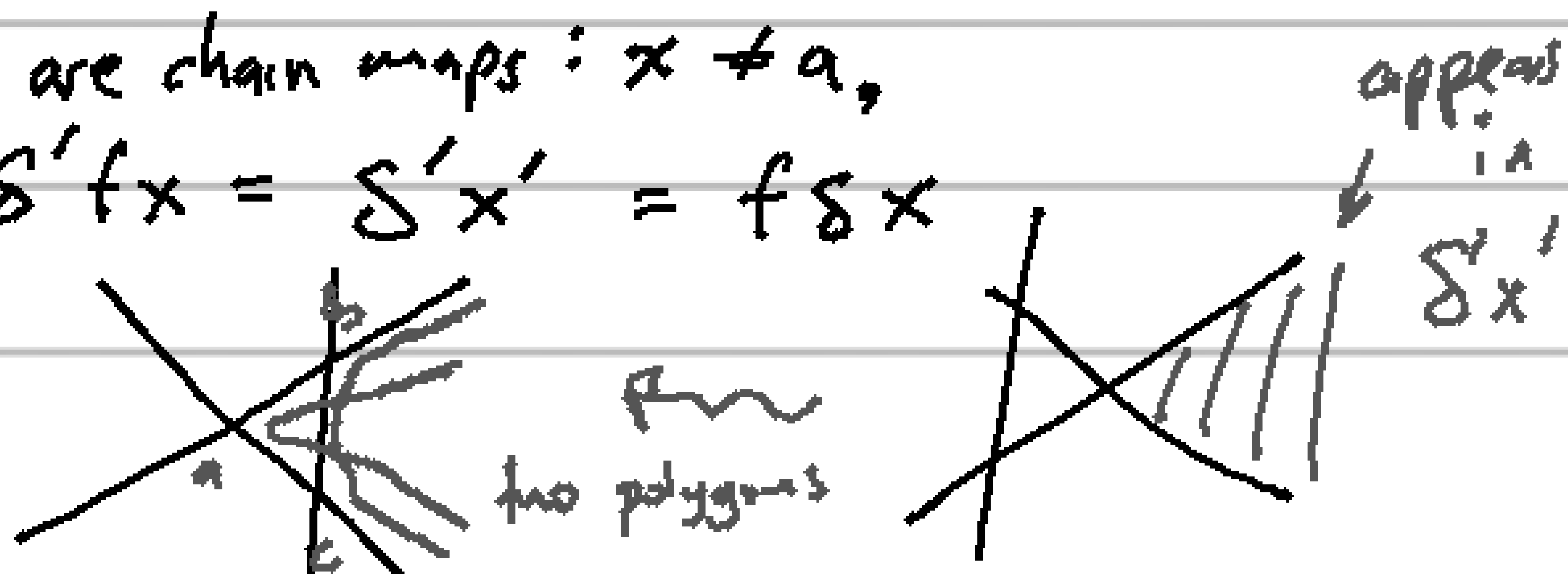
$f : a \longmapsto a' + b'c'$

$\quad x \longmapsto x'$ for all other x.

$$\left.\begin{array}{l} a+bc \longleftarrow a' \\ x \longleftarrow x' \end{array}\right\} : g$$

f, g are chain maps: $x \neq a$,

$$\delta' f x = \delta' x' = f \delta x$$



$\Rightarrow$ chain map $\Rightarrow$ A_∞ morphism.

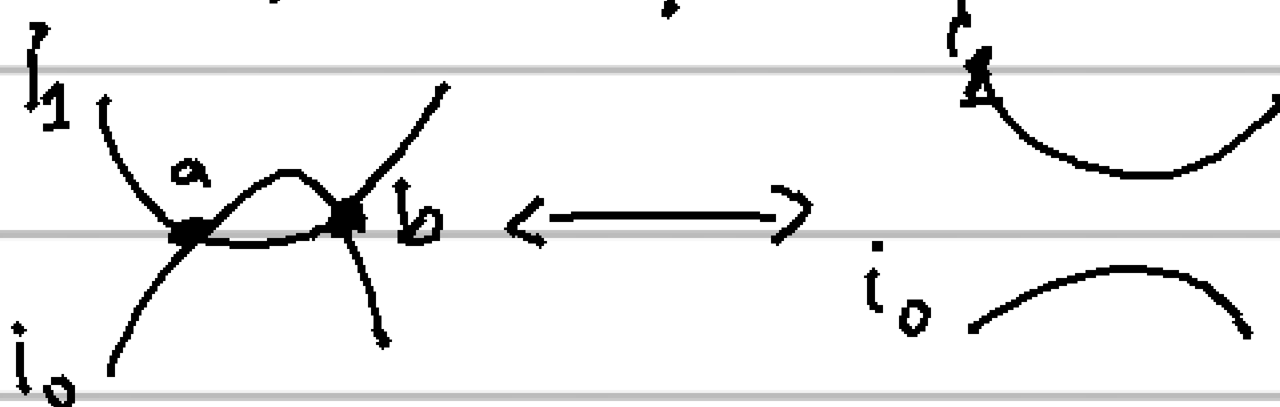
$\Rightarrow$ f, g are strict inverse A_∞ isos.

For $x = a$, same sort of argument.

This case: strict iso.

For move II, we need a quasi-iso.

II.



$$\delta a = b + v$$

Define f:
$$a \longmapsto 0$$
$$b \longmapsto v'$$
$$x \longmapsto x' \text{ all other } x.$$

Let $h : T_1 \to T_1[1]$:

$h_1(b) = a$

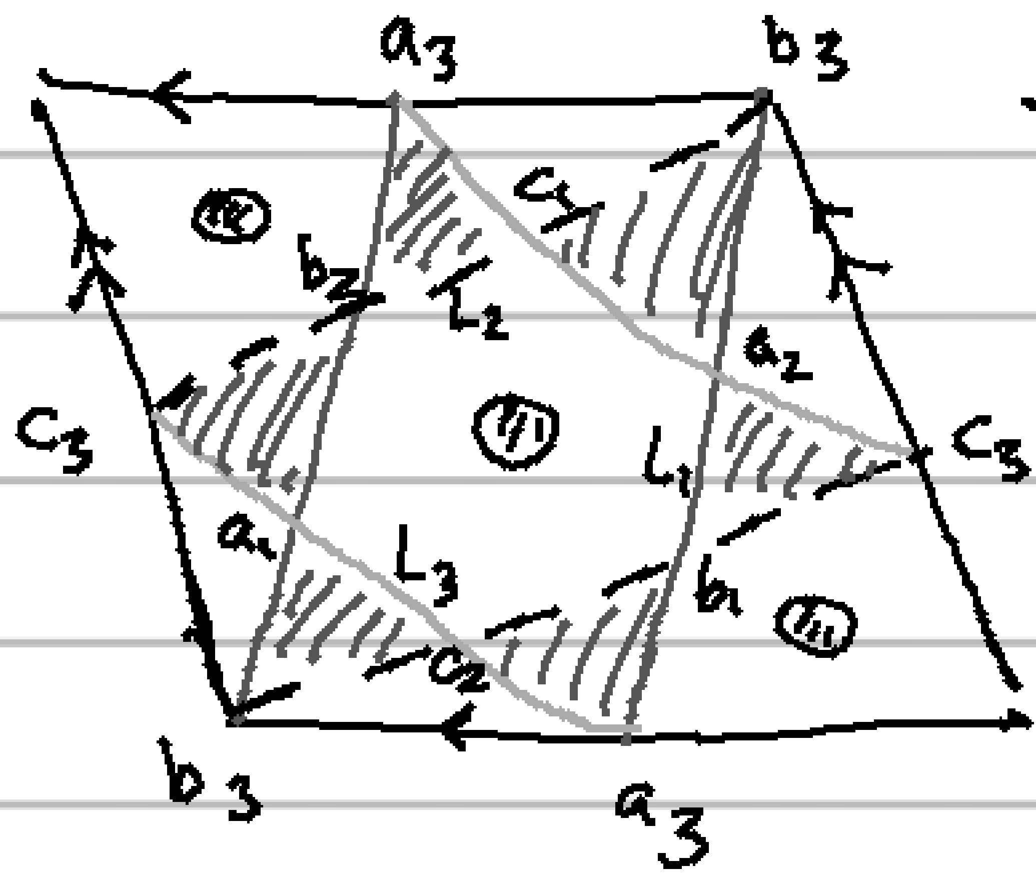
$h_1(x) = 0$ else $g_1 : T_1' \longrightarrow T_1$

$g_1(x') = x + h_1 \delta_1 x$.

f is an A_∞-morphism, $f_1 g_1 = \mathrm{id}$, $g_1 f_1 = \mathrm{id} + h\delta_1 + \delta_1 h_1$ $\Rightarrow$ By Perturbation lemma, can extend g to $g : T' \to T$ s.t.

$f \circ g$, $g \circ f$ homotopic to id.

Quick Example:



Torus w/ 3 points removed.

$$a_i b_j = \varepsilon_{ijk} c_k$$

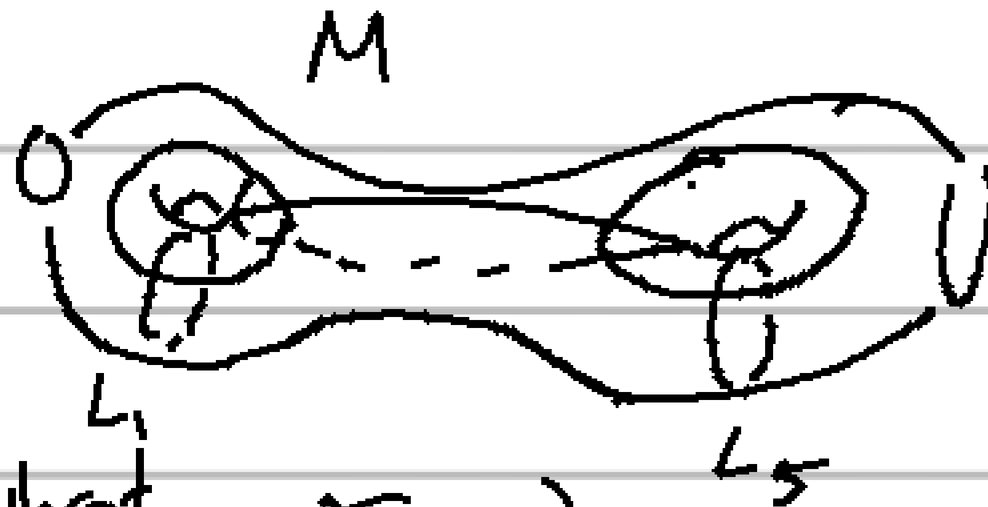
Get:

$$\begin{tikzcd} L_1 \arrow[r, "V"] \arrow[rr, bend right, "\Lambda^2 V"'] & L_2 \arrow[r, "V"] & L_3 \end{tikzcd}$$

$V \cong \mathbb{C}^3$.

From Paul: Warning! Just b/c $\mathcal{F}^{\to}$ is combinatorial doesn't mean $\mathcal{F}$ is!

Ex:



full subcat w/ obj. $L_1, \dots, L_5$ $\subset \mathcal{F}(M)$

Thm: This is not formal.

(even though there are no polygons in picture above. when you perturb, get small polygons