Day 3 Talk 1: Carl, Quantum Cohomology

I. J-holom. spheres

(M,ω) $2n$-dim'l.

$\pi_2(M) \longrightarrow H_2(M) \ni A$ Hurewicz

image are called "spherical classes"

Thm: $\exists$ a subset $\mathcal{J}_{reg}(A) \subset J_\tau(M,\omega)$ of 2nd category s.t. $\forall J \in \mathcal{J}_{reg}$

$$\mathcal{M}^*(A;J) = \left\{ u: S^2 \xrightarrow[\text{simple}]{} M \;\middle|\; \begin{array}{l} u \text{ J-holom.} \\ [u] = A \end{array} \right\}$$

is a smooth mfold of dim $= 2n + 2c_1(A)$

First note: $PSL_2(\mathbb{C}) \circlearrowright \mathcal{M}^*(A;J)$, & this is non-compact, so our moduli space is non-compact.

Def: $\mathcal{M}^*(A;J) \times_{PSL_2} (\mathbb{C}P^1)^k = \mathcal{M}^*_{0,k}(A,J)$

$$\begin{array}{ccc} \mathcal{M}^*(A;J) \times_{PSL_2} (\mathbb{C}P^1)^k & & \mathcal{M}^*_{0,k}(A,J) \\ \downarrow ev & & \uparrow \dim \\ M^k & & 2n + 2c_1(A) + 2k - 6 \end{array}$$

Def: (M,ω) is semi-positive if $\forall A \in H_2(M)^{sph}$

if $\omega(A) > 0$, $c_1(A) \geq 3 - n \Rightarrow c_1(A) \geq 0$.

Thm: (M,ω) semi-pos. then $\exists\, \mathcal{J}_{reg}(M,\omega) \subset J_\tau(M,\omega)$ of 2nd category s.t. $\forall A \in H_2$ w/ $c_1(A) > 0$, $J \in \mathcal{J}_{reg}(M,\omega)$

then $ev : \mathcal{M}^*_{0,k}(A;J) \longrightarrow M^k$

gives a pseudocycle of dimension $\underbrace{2n+2c_1+2k-6}_{= d}$,

i.e. a smooth map $f : V^d \longrightarrow M$

s.t. $\dim(\overline{f(V)} - f(V)) \leq d-2$.

smooth mfold, dim d.

II. 3 point GW invt.

Notation: $H^k(X) =$ free part of $H^k(X;\mathbb{Z})$

$H^k(X) = \mathrm{Hom}(H_k(X;\mathbb{Z}), \mathbb{Z})$

$a \in H^i(M)$, $b \in H^j(M)$, $c \in H^k(M)$, $A \in H_2^{sph}(M)$

$GW^3_A(a,b,c)$

$= ev \cdot (\alpha \times \beta \times \gamma)$ $(\alpha = PD(a) \cdots)$.

$\dim(\mathcal{M}^k_{0,3}) = 2n + 2c_1(A)$

Need $\deg a + \deg b + \deg c = 2n + 2c_1(A)$

Rmk: GW^3_A is graded commutative in a, b, c

(exercise on orientations).

e.g. $A = 0$ const maps, so $ev = [\Delta M] \in H_{2n}(M^3)$

$GW^3_0(a_1, a_2, a_3) = [\Delta M] \cdot \alpha = \int_M a_1 \cup a_2 \cup a_3$

Ex. $(\mathbb{CP}^n, \omega_0)$, $H_2(\mathbb{CP}^n) = \mathbb{Z} = \langle L \rangle \leftarrow [\mathbb{CP}^1]$

HW: $c_1(L) = n+1$

$GW^3_{mL}(a,b,c)$ doesn't vanish only when

$\sum \deg = 2n + 2m(n+1)$, i.e

vanish unless: $m=0$, $\sum \deg = 2n$

$m=1$, $\sum \deg = 4n+2$

Remains to compute

$$GW^3_L(p^i, p^j, p^k) = 1 \iff i+j+k = 2n+1$$

$p = PD[L]$ $\quad GW_L(p, p^n, p^n) = 1.$

conclusion:

$$GW^3_{mL}(p^i, p^j, p^k) = \begin{cases} 1 & m=0,\ i+j+k=n \\ 1 & m=1,\ i+j+k=2n+1 \\ 0 & \text{otherwise} \end{cases}$$

(point is that $i+j+k = 2n+2m(n+1) \geq 6n+1$)

III. Quantum Cohomology

Assume M is monotone ($\omega(A) = \lambda c_1(A)$, $\lambda > 0$)

As abelian groups, $QH^*(M) = H^*(M) \otimes \mathbb{Z}[q, q^{-1}]$

$N = \min\limits_{A,\, c_1(A) \neq 0} |c_1(A)|$ $\quad \deg q = 2N.$

Ring structure:

$$a \in H^k(M),\ b \in H^l(M),$$

$$a * b = \sum_{A \in H_2} (a*b)_A \, q^{c_1(A)/N}$$

$$(a*b)_A \in H^{k+l-2c_1(A)}$$

Define $(a*b)_A$ to satisfy

$$\langle (a*b)_A, c \rangle = GW^3_A(a,b,c)$$

$$\langle x,y \rangle := \int_M x \cup y \qquad 2n + 2c_1(A) = \sum \deg$$

$$\Rightarrow 0 \le c_1(A) \le 2n$$

$\Rightarrow$ sum is finite.

Extend similarly to

$$QH^*: QH^* \otimes QH^* \longrightarrow QH^*$$

Rmks: 1) this is distributive

2) " " graded commutative.

3) Thm: Is associative.

Back to $\mathbb{P}^n$:

$$p^i * p^j = \sum_m (p^i * p^j)_{mL} \, q^{c_1(mL)/(n+1)} \quad q^m$$

$$\langle (p^i * p^j)_{mL}, p^k \rangle = GW^3_{mL}(p^i, p^j, p^k)$$

$$p^i * p^j = \begin{cases} p^{i+j} & \text{if } i+j \le n. \\ p^{i+j-2n-1} \cdot q & \text{if } n \le i+j \le 2n \end{cases}$$

← not quite right.

So, $QH^*(P^n) = \mathbb{Z}[p, q^{\pm}] \big/ (p^{n+1} = q)$

<u>Sketch of associativity:</u>

$$QH^* \otimes QH^* \otimes QH^* \longrightarrow QH^*$$
$$a \otimes b \otimes c \longmapsto (a*b)*c.$$

$$(a*b)*c = \pm (b*c)*a = a*(b*c)$$

$$\Updownarrow$$

$((a*b)*c)_A$ is graded commutative.

To do so, pair:

$$\langle ((a*b)*c)_A, d \rangle = \Big\langle \Big\langle \sum_B (a*b)_B\, q^{\omega(B)/N} * c \Big\rangle_A, d \Big\rangle$$

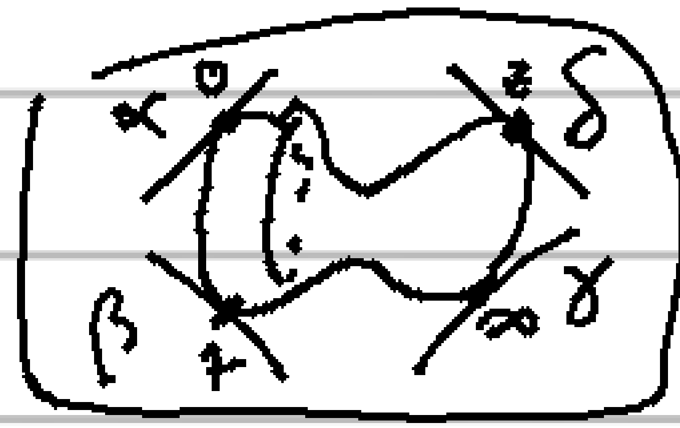
$$= \Big\langle \sum_B ((a*b)_B * c)_{A-B}, d \Big\rangle$$

$$= \sum_B GW^3_{A-B}((a*b)_B, c, d).$$

$= \pm \Big\{$  $\Big\}$

$$= \sum_{B} GW^{2,2}_{B,A-B}(a,b;c,d)$$

$$= GW_A^{(0,1,\infty,z)}(a,b,c,d)$$

$$\# \Big\{$$  $$\Big\}$$

This guy is graded commutative, so we're done.

Coeff: (M,ω) any closed symp., define

Λ_ω Novikov ring of ω.

formal sums $\lambda = \sum_{A \in H_2} \lambda(A) e^A$

s.t. $\# \left\{ A \in H_2(M) \;\middle|\; \begin{array}{l} \lambda(A) \neq 0 \\ \omega(A) \le c \end{array} \right\} < \infty \quad \forall c \in \mathbb{R}$

$$a * b = \sum_A (a*b)_A e^A$$

If (M,ω) is CY, i.e. $c_1 = 0$ on spherical classes, can use the universal Novikov ring

$$\Lambda^0 = \left\{ \lambda = \sum_{\varepsilon \in \mathbb{R}} \lambda_\varepsilon t^\varepsilon \;\middle|\; \#\left(\varepsilon \in \mathbb{R} \mid \lambda_\varepsilon \neq 0 \;\; \varepsilon \leq c\right) < \infty \;\; \forall \varepsilon \in \mathbb{R} \right\}.$$

$$a * b = \sum_A (a*b)_A t^{\omega(A)}$$

for general case:

$$\Lambda = \Lambda^0[q, q^{-1}],$$

$$a * b = \sum_A (a*b)_A t^{\omega(A)} q^{c_1(A)}.$$