Day 3 Talk 2: Maksim

Monotone Lagrangians

$L \subset M$, $\mu : \pi_2(M, L) \to \mathbb{Z}$ relative c_1

$\omega : \pi_2(M, L) \to \mathbb{R}$ $\quad \int_D u^*\omega$.

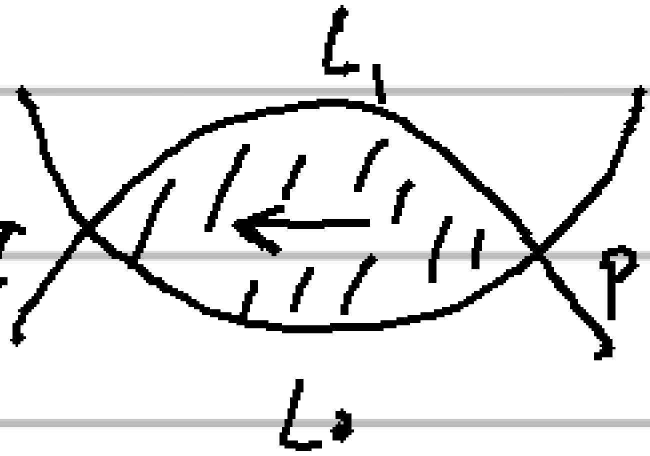
Monotone means $\mu = \lambda \omega$, $\lambda > 0$.

Coeff ring: $\Lambda = \left\{ \sum a_k t^{\nu_k} \;\middle|\; \nu_k \to \infty,\ a_k \in \mathbb{Z}, \mathbb{C} \right\}$

Generalization: flat line bundles

Ob. (L, u) $\quad u$ — flat $U(1)$ bundle on L.

$$CF((L_0, u_0), (L_1, u_1)) = \bigoplus_{p \in L_0 \cap L_1} \mathrm{Hom}(u_{0p}, u_{1p}) \otimes \Lambda \langle p \rangle$$



$$\partial p = t^{\int u^*\omega} \underbrace{e^{i\theta}} \langle q \rangle$$

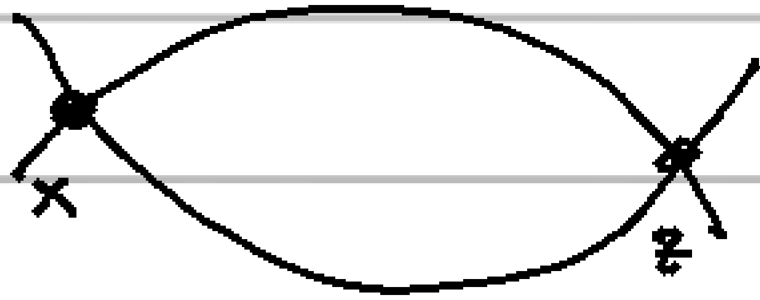
$h \in \mathrm{Hom}(u_{0p}, u_{1p})$

$f^\vee$

$$\Big(h^{-1} \big(\mathrm{hol}_{q \to p}\, u_1\, f\big)\, \mathrm{hol}_{p \to q}\, u_0\Big) : u_{0p} \to u_{0p}$$

rotation by θ

Idea of $\delta^2 = 0$.



$\langle \delta^2 x, z \rangle$ — bdry points of 1-dim'l moduli space $\mathcal{M}(x,z)/\mathbb{R}$

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$\hat{\mathcal{M}}(x,z)$

$\mu(\beta) = 2$

In the $\partial \mathcal{M}_\beta(x,z) \longrightarrow$ strips, disc bubbles, sphere bubbles.

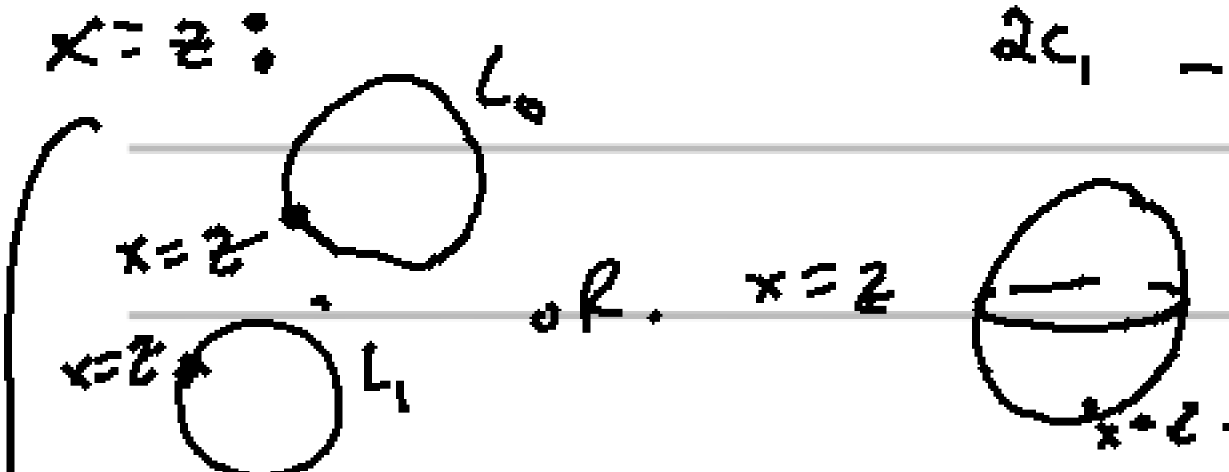
$\mathcal{M}$ is at least 2

$x \neq z$ — at least one strip.

all bubbles will have to have $\omega = 1$ — no disks ($\mu \geq 2$).

$2c_1$ — no spheres ($2c_1 \geq 2$).

$x = z$:

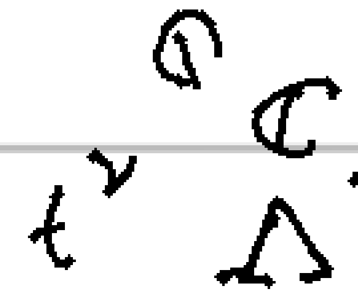


$c_1(\text{sphere}) = 1$

$2n-2$ dim'l geometric diagram, misses points!

$2n + 2 - 6 = 2n - 4$, so no sphere bubbles (generic argument)

Define $m_0(L) = \sum_{\substack{\beta \\ \mu(\beta)=2}}$ # discs of index 2 through a given point with $(\nabla_{\partial\beta})$ factor.



m_0 is a complex # associated to L.

$$0 = \partial\hat{M} = \langle \delta^2 x, z \rangle + m_0(L_0) - m_0(L_1)$$

Point: For each $\lambda \in \mathbb{C}$, we get a Fuk with objects Lagrangians w/ $m_0(L_0) = m_0(L_0) = \lambda$.
(m_0 called "central charge" or really just a charge).

Thm: There is an action of $QH(X)$ on $HF(L,L)$

Use: Morse-Bott picture.

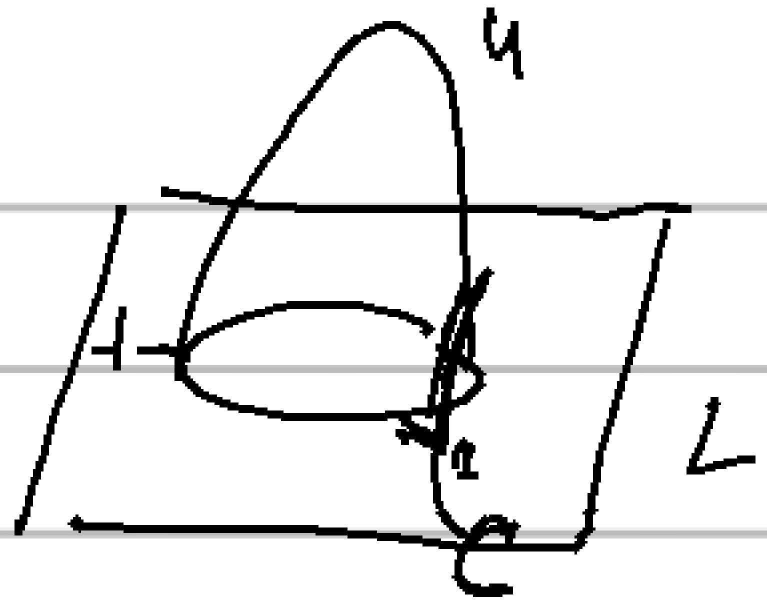
CF geometric singular chains on L

$$\delta = \partial + \delta'$$

$$\delta C = \partial C + \sum_{\beta \in \pi_2(X,L)} \delta'_\beta C$$

M_β = moduli space of 2 pointed hol. maps.

See picture



$$\delta_\beta^1 c = ev_{-1*}((ev_1)^*(c))$$

$$n + \mu(\beta) + 2 - 3 = n + \mu - 1.$$

$(2n + 2c_1)$

$(\dim(\delta_\beta' c) = \dim C + (\mu - 1))$

(M_β, L_1) — 1 pointed, with

$$ev_{1*}[[(M_{\beta, L, 1})]] \in C_*(L)$$

$\shortparallel$

$m_0[L]$

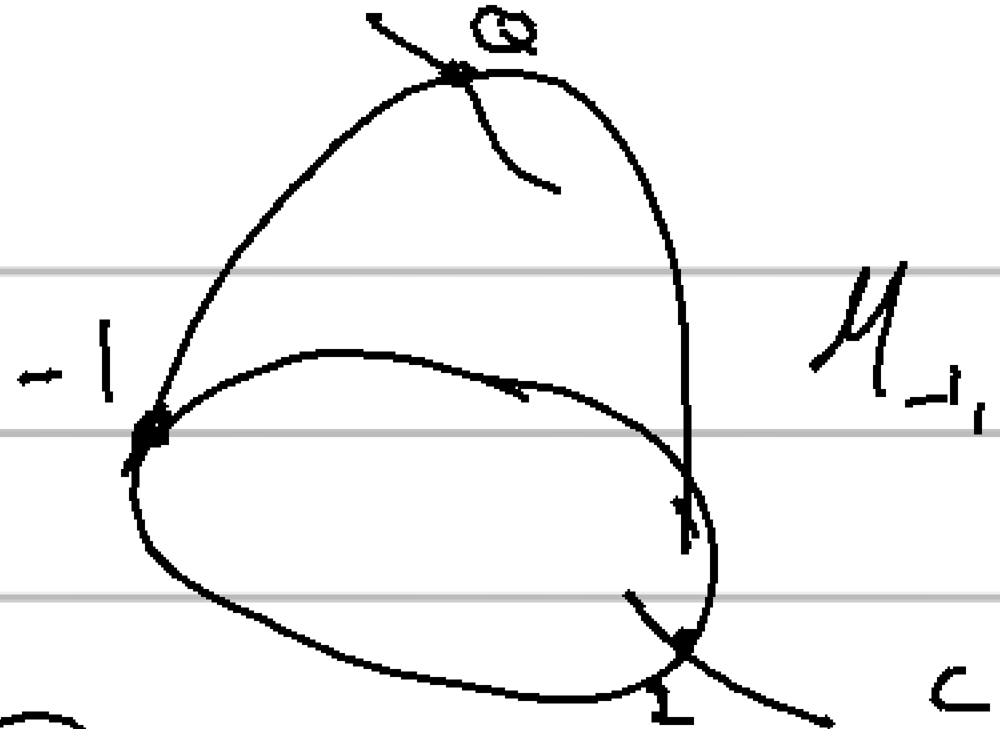
m_2 3 pointed discs

$$m_2(c_1, c_2) = \sum_\beta$$

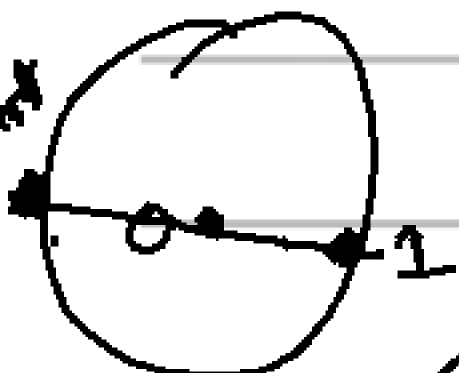
$$ev_{3*}(ev_1^* \times ev_2^* c_1 \times c_2)$$

$Q \in C_*(M)$, $C \in C_*(L)$

$Q \cap_\beta C$



$\mathcal{M}_{-1,0,1}(\beta, L)$



$ev_{-1*}(ev_0 \times ev_1)[Q \times C]$

$\underline{Q \cap_\beta C}$ & $Q \cap C = \sum_\beta Q \cap_\beta C$.

$\delta(Q \cap C) = \pm (\partial Q) \cap C \pm Q \cap \delta C$

* extra terms involving $m_0 = m_0$ (L)

$Q \cap \partial C$.

$\underline{\text{Fano}}$: $-K_X$ is ample.
$[D]$

$\underline{\text{Ex}}$: $\mathbb{CP}^n$, $c_1 = (n+1)P$

D = union of coordinate $\mathbb{CP}^{n-1}$;
$z_i = 0$, $i = 0, \dots, n$.

On $X \setminus D$, we have a holom. vol. form.

<u>Thm</u>: If m_0 is not an eigenvalue of $*c_1(X): QH \to QH$ then $HF(L,L) = 0$.

i.e. this theorem is about "charge quantization" (i.e. most Lagrangians have zero HF).

Two ingredients:

- $[c_1(X)] \cap [L] = m_0 [L]$.

$$\underset{QH}{\underset{\Uparrow}{[c_1(X)]}} \cap \underset{HF}{\underset{\Uparrow}{[L]}}$$

$(c_1 - m) \cap [L]$

(also know $Q_1 \cap (Q_2 \cap C) = (Q_1 * Q_2) \cap C$).

If $(c_1 - m)*$ is invertible, then

$$\underset{\overset{\shortparallel}{L}}{[x] \cap L} = \alpha * ((c_1 - m_0) \cap L) = \alpha * 0 = 0.$$

ex: $\mathbb{C}P^2$

$$\begin{array}{c|ccc} & 1 & p & p^2 \\ 1 & 0 & 1 & 0 \\ p & 0 & 0 & 1 \\ p^2 & q & 0 & 0 \end{array} \quad \text{— } * p$$

$QH = \dfrac{\mathbb{C}[p,q]}{p^{n+1} = q}$

eigenvalues

$c_1 = 3p$ char. pol.: $\lambda^3 - q$.

eig. values are $\underset{m_0}{3q^{\frac{1}{3}}}$, $3\sqrt[3]{1}\, q^{\frac{1}{3}}$, $3(\sqrt[3]{1})^2 q^{1/3}$

Clifford torus: $S^1 \times S^1 \times S^1 \subset \mathbb{C}^3$

$T^2 \subset \mathbb{C}P^2$

classes of 3 disks, which together give a sphere

Each disk has area $1/3$.

Contributions $q^{1/3}, q^{1/3}, q^{1/3}$, together $3q^{1/3}$, corresponding to eigenvalue.

If we change local system on T^2, so that holonomies are $\sqrt[3]{1}$, get a different eigenvalue

Conclusion:

each eigenvalue, get a clifford torus w/ corresponding holonomies

$(1, 1, 1)$
(ζ, ζ, ζ)
$(\zeta^2, \zeta^2, \zeta^2)$

Unwritten result: these objects split generate the relevant Fukaya category.

(In Fano case, $\mu(\beta)$ given by $\beta \cdot [D]$ or something, that's what helps.)

Paul: This doesn't use Fano case! There are some issues w/ non-orientability.

CF: Auroux, Mirror Symm & T-Duality