Day 3 Talk 4:

Sarah, Derived Categories

Defn: An A_∞ category (strictly unital) $\mathcal{A}$ is a set of objects $\mathrm{Ob}\,\mathcal{A}$, graded v spaces $\hom_{\mathcal{A}}(X_0, X_1)$ for any pair of objects X_0, X_1 + composition maps $\forall d \geq 1$:

$$\mu_{\mathcal{A}}^d : \hom(X_{d-1}, X_d) \otimes \cdots \otimes \hom(X_0, X_1) \to \hom(X_0, X_d)[2-d]$$

& satisfy the usual A_∞ equations.

Additionally, $\exists\, e_X \in \hom(X,X)$ (not nec. unique) s.t.

1) $\mu_{\mathcal{A}}^1(e_X) = 0$

2) $\pm\mu_{\mathcal{A}}^2(e_X, a) = a = \mu_{\mathcal{A}}^2(a, e_X)$ $\forall a \in \hom_{\mathcal{A}}(X_0, X_1)$

$\Rightarrow$ uniqueness

3) $\mu^d(a_{d-1}, \dots, e_{X_i}, \dots, a_1) = 0$ $\forall d \geq 2$, $a_k \in \hom(X_{k-1}, X_k)$ $\forall i$.

Def'n: For $\mathcal{A}$ an A_∞-category,

$H(\mathcal{A})$ — the cohom. cat. associated to $\mathcal{A}$

* $\mathrm{Ob}\, H(\mathcal{A}) = \mathrm{Ob}(\mathcal{A})$
* $H(\hom(X_0, X_1), \mu^1_{\mathcal{A}}) = \mathrm{Hom}_{H(\mathcal{A})}(X_0, X_1)$

Assoc. for $d=1 \Rightarrow (\mu^1)^2 = 0$

* discard all homs of deg $\neq 0$, $H^0(\mathcal{A})$.

Want to define $D(\mathcal{A})$, $\mathcal{A}$ an A_∞ category.

Old	A_∞
$\mathcal{A}$ abelian	$\mathcal{A}$ A_∞-cat.
$C^*(\mathcal{A}), K^*(\mathcal{A})$	$\mathrm{Tw}\,\mathcal{A}$ twisted complexes
$D(\mathcal{A})$	$H^0(\mathrm{Tw}\,\mathcal{A}) = D(\mathcal{A})$

triangulated

Def'n: An A_∞ functor between $\mathbb{Z}$-A_∞ categories

$$\mathcal{F} : \mathcal{A} \to \mathcal{B}$$

* map on sets $\mathcal{F}^0 : \mathrm{ob}\,\mathcal{A} \to \mathrm{ob}\,\mathcal{B}$
* $\forall\, d \geq 1$,

$$\mathcal{F}^d : \hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0, X_1) \longrightarrow \hom_{\mathcal{B}}(\mathcal{F}X_0, \mathcal{F}X_d)[1-d]$$

* The $\mathcal{F}^d$ satisfy relations w.r.t. μ^d

  * $\mathcal{F}^1(e_x) = e_{\mathcal{F}(x)}$

  * $\mathcal{F}^d(a_{d-1}, \dots, e_{x_i}, \dots, a_1) = 0$

In particular, $\forall\, d \geq 2$, any n.

$\mu^1 \mathcal{F}^1 = \mathcal{F}^1 \mu^1$.

For $\mathcal{F} : \mathcal{A} \to \mathcal{B}$,

$H(\mathcal{F}) : H(\mathcal{A}) \longrightarrow H(\mathcal{B})$ (or $H^0 \to H^0$)

For $[a]$, $H^0(\mathcal{F})[a] = [\mathcal{F}^1(a)]$

• Say $\mathcal{F}$ is full/faithful/quasi-eq. if $H(\mathcal{F})$ is full/faithful/quasi-eq.

<u>Thm</u>: $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is a quasi-eq. $\Longrightarrow$ $\exists\ \mathcal{G} : \mathcal{B} \to \mathcal{A}$ ~~a~~ quasi-eq. s.t.

$\mathcal{F} \circ \mathcal{G} = \mathrm{Id}_{\mathcal{B}}$, $\mathcal{G} \circ \mathcal{F} = \mathrm{Id}_{\mathcal{A}}$.

<u>Ex</u>: $\mathcal{T}_2$:
$$\begin{tikzcd} \underset{x_0}{\bullet} \arrow[loop left, "e_{x_0}"] \arrow[r, bend left, "\phi_0"] & \bullet \arrow[loop right, "e_{x_1}"] \arrow[l, bend left, "\phi_1"] \end{tikzcd}$$

$\mu^2(\phi_1, e_{x_1}) = \phi_1$

$\mu^2(e_{x_0}, \phi_1) = \phi_1$, etc.

$\mu^2(\phi_1, \phi_0) = e_{x_0}$

& vice versa.

Exercise: Check it's A_∞

Def'n: $X_0, X_1 \in \mathcal{A}$, A_∞ are isomorphic if $\exists$ A_∞ functor $\mathcal{F}: T_2 \to \mathcal{A}$

$$x_0 \mapsto X_0$$
$$x_1 \mapsto X_1$$

* In $H(\mathcal{A})$,

$X_0 \simeq X_1$.

$$T_3: \begin{tikzcd} x_0 \arrow[loop left] \arrow[rr, "\phi_0 \,\}\, \deg 0"] & & x_1 \arrow[loop right] \arrow[dl, "\phi_1 \,\}\, \deg 0"] \\ & x_2 \arrow[loop below] \arrow[ul, "{[1]}", "\phi_2 \,\}\, \deg 1"'] & \end{tikzcd}$$

$\mu^3(\phi_2, \phi_1, \phi_0) = e_0$

$\mu^3(\phi_0, \phi_2, \phi_1) = e_1$

$\mu^3(\phi_1, \phi_0, \phi_2) = e_2$

means:

$\hom(x_0, x_1) = k\phi_0$

$\hom(x_0, x_2) = 0$

$\hom(x_2, x_0) = k\phi_2 \,\}\, \deg{-1}$.

Def'n: An *exact triangle* in $\mathcal{A}$ is the image of $\mathcal{F} : T_3 \to \mathcal{A}$

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* If $\mathcal{G} : \mathcal{A} \to \mathcal{B}$ is any functor, $\mathcal{F} : T_3 \to \mathcal{A}$ is exact Δ, then $\mathcal{G} \circ \mathcal{F} : T_3 \to \mathcal{B}$ is an exact Δ in $\mathcal{B}$.

If

$$\begin{tikzcd} X_0 \arrow[r] & X_1 \arrow[dl] \\ X_2 \arrow[u, "{[1]}"] & \end{tikzcd}$$

is an exact Δ in $\mathcal{A}$, $X \in \mathcal{A}$, then $\exists$ long exact sequence

$$\mathrm{Hom}_{H(\mathcal{A})}(X, X_0) \to \mathrm{Hom}_{H(\mathcal{A})}(X, X_1) \to \mathrm{Hom}_{H(\mathcal{A})}(X, X_2) \to$$

i.e. exact:

$$\to \mathrm{Hom}^d(X, X_0) \to \mathrm{Hom}^d(X, X_1) \to \mathrm{Hom}^d(X, X_2) \to \cdots \to \mathrm{Hom}^{d+1}(X, X_0) \to \cdots$$

& exact

$$H(X_2, X) \to \mathrm{Hom}(X_1, X) \to \mathrm{Hom}(X_0, X).$$

* other A_∞-analogues of Δ-axioms which these Δ's satisfy.

(can, e.g. derive octahedral axiom very easily).

Def'n: An A_∞-cat. $\mathcal{A}$ is triangulated if it's "closed under shifts" and every morphism can be completed to an exact triangle.

Tw $\mathcal{A}$: **Def'n**: A twisted complex in $\mathcal{A}$ is a pair (X, δ_X)

* $X = \bigoplus_{i \in I} V^i \otimes X^i$ — V^i vector spaces fin. dim'l, $X^i \in \mathrm{Ob}\,\mathcal{A}$, $|I| < \infty$

* $\delta_X \in \hom^1(X, X)$ s.t. $\underbrace{\sum_{d \geq 1} \mu^d(\delta_X, \dots, \delta_X) = 0}_{\text{gen. Maurer-Cartan.}}$

technical condition: I ordered set, & δ_X decreasing

$$\hom_{\mathrm{Tw}\mathcal{A}}(\underset{V \otimes X}{\underset{\shortparallel}{X}}, \underset{W \otimes Y}{\underset{\shortparallel}{Y}}) = \bigoplus \mathrm{Hom}(V^i, W^j) \otimes \hom_{\mathcal{A}}(X^i, Y^j).$$

(when $\mathcal{A}$ is already triangulated, this gives a quasi-eq. category).

$$\mu^d_{\mathrm{Tw}\mathcal{A}}(a_d, \dots, a_1) = \sum_{i_0, \dots, i_d} \mu^{d + i_0 + \dots + i_d}(\underbrace{\delta_{X_d}, \dots, \delta_{X_d}}_{i_d}, a_d, \delta_{X_{d-1}}, \dots, \delta_{X_{d-1}}, \dots, a_1, \underbrace{\delta_{X_0}, \dots, \delta_{X_0}}_{i_0}).$$

"twisting μ^d by δ"

Fact: $Tw\,\mathcal{A}$ is a triangulated category.

Def'n: $D(\mathcal{A}) = H^0(Tw\,\mathcal{A})$

* Triangulated category in "old" sense.

Shifts: $X = \bigoplus_{i \in I} V^i \otimes X^i$

$$SX = X[1] = \bigoplus_{i \in I} V^i[1] \otimes X^i$$

$$SX = \delta_X[1]$$

Triangles $\longleftrightarrow$ Define cones of morphisms.

$f : X \to Y$, $\mu^1(f) = 0$

$$\mathrm{Cone}(f) = \left(X[1] \oplus Y, \begin{pmatrix} \delta_X[1] & 0 \\ -f & \delta_Y \end{pmatrix} \right).$$

Fact:
$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \arrow[dl] \\ & \mathrm{Cone}(f) \arrow[ul, "{[1]}"] & \end{tikzcd}$$

Ex: $\mathcal{A} := \underset{x}{\bullet} \circlearrowleft e_x \qquad \mu^2(e_x, e_x) = e_x$

$Tw\,\mathcal{A} \ni X = V \otimes x$, graded vector space

δ_X - differential on V

$Tw\,\mathcal{A} = dg\text{-}Vect_k$

$\eta^1_{\mathrm{Tw}A}(f) = \delta_x f \pm f\delta_x$

Homomorphisms in $H(\mathrm{Tw}\mathcal{A})$ will be homotopy equivalence classes of chain maps.

$$\Rightarrow H^0(\mathrm{Tw}\mathcal{A}) = K^\bullet(\mathrm{Vect}_k)$$
$$\qquad\qquad\qquad\qquad \parallel$$
$$\qquad\qquad\qquad\qquad D(\mathrm{Vect}_k).$$