Day 4, Talk 1: Parker, Exceptional Collections

Ex: $D^b(\mathbb{P}^1)$ ← derived category associated to Abelian cat. $Coh(\mathbb{P}^1)$.

$Id: D^b(\mathbb{P}^1) \to D^b(\mathbb{P}^1)$ equivalent to using

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, "\Delta"] & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^1 & & \mathbb{P}^1 \end{tikzcd}$$

want to get rid of derived tensor!

$$Id \cong R\pi_{2*}(\mathcal{O}_\Delta \otimes^L \pi_1^* -)$$

$$G: 0 \to \underset{-1}{\mathcal{O}(-1) \boxtimes \mathcal{O}(-1)} \to \underset{0}{\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}} \to 0$$

$$\Phi_{G^{-1}}(F) \cong R\pi_{2*}(\pi_2^* \mathcal{O}(-1) \otimes \pi_1^* \mathcal{O}(-1) \otimes \pi_1^* F)$$

$\cong R\pi_{2*}(\pi_2^* \mathcal{O}(-1) \otimes \pi_1^* F(-1))$, do some more work...

$$\cong R\Gamma(F(-1)) \otimes \mathcal{O}(-1) \cong R\mathrm{Hom}(\mathcal{O}(1), F) \otimes \mathcal{O}(-1)$$

$$\Phi_{G^0}(F) = R\Gamma(F) \otimes \mathcal{O} \cong R\mathrm{Hom}(\mathcal{O}, F) \otimes \mathcal{O}_{\mathbb{P}^1}$$

Namely,

$$\mathrm{Id} \cong \Phi_G(\mathcal{F}) =$$

$$\mathrm{Cone}\Big(R\mathrm{Hom}(\mathcal{O}(1), \mathcal{F}) \underset{-1}{\otimes} \mathcal{O}_{\mathbb{P}}(-1) \to R\mathrm{Hom}(\mathcal{O}, \mathcal{F}) \otimes \mathcal{O} \Big).$$

If $\mathcal{A}$ = extension category of $\mathcal{O}$, $\mathcal{O}(-1)[1]$

$$D^b(\mathbb{P}^1) \cong D^b(\mathcal{A}).$$

Quiver picture:

$$\mathcal{O}(-1)[1] \, \bullet \rightrightarrows \bullet \, \mathcal{O}.$$

"Kronecker quiver"

(Jordan normal form then gives us torsion sheaves on the plane. —)

<u>Def'n</u>: An <u>exceptional collection</u> of a triangulated cat T consists of the following:

- I an ordered (finite) set

$\{Y_i\}_{i \in I}$ $Y_i \in T$

$$\text{s.t. } R\mathrm{Hom}^k(Y_i, Y_j) = \begin{cases} 0 & \text{if } i > j \\ \mathbb{K} & \text{if } i = j \ \& \ k = 0 \\ 0 & \text{if } i = j,\ k \neq 0 \\ \text{fin. dim'l vect-space} & \text{otherwise} \end{cases}$$

Definition: An exceptional collection $\{Y_i\}_{i \in I}$ is full if $T(\{Y_i\}) \cong T$ where $T(Y_i)$ is the triangulated hull.

e.g. $\{\mathcal{O}(-1)[1], \mathcal{O}\}$ is a full exceptional collection of $D^b(\mathbb{P}^1)$.

Ex: $\{\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(-1)\}$ vs. $\{\mathcal{O}(-1)[1], \mathcal{O}\}$

Def'n: left mutation of an object Y by an object X

$$L_X Y = \mathrm{Cone}\left(\bigoplus_i \mathrm{Hom}(X[-i], Y) \otimes X[i] \xrightarrow{ev} Y\right\}$$

this is only an isomorphism class, but pick one, it doesn't matter.

((Cone is like a mixed kernel & cokernel).

Def'n: A left mutation by the i^{th} object on an exceptional collection $\{Y_i\}$ is the following collection:

$$\begin{tikzcd}[column sep=small]
Y_0 \arrow[d] & Y_1 \arrow[d] & \cdots & Y_{i-1} \arrow[dr] & Y_i \arrow[dl] & Y_{i+1} & \cdots & Y_n \arrow[d] \\
Y_0 & Y_1 & \cdots & L_{Y_{i-1}} Y_i & Y_i & Y_{i+1} & \cdots & Y_n
\end{tikzcd}$$

Results: (If (A,B) are an exceptional pair)

1) $\exists$ the concept of a right mutation
2) $R\mathrm{Hom}(A, L_A B) = 0$
3) $R_A L_A B \cong B$
4) Given exc. coll. (A, B, C)

$$L_{L_A B} L_A C \cong L_A(L_B C)$$

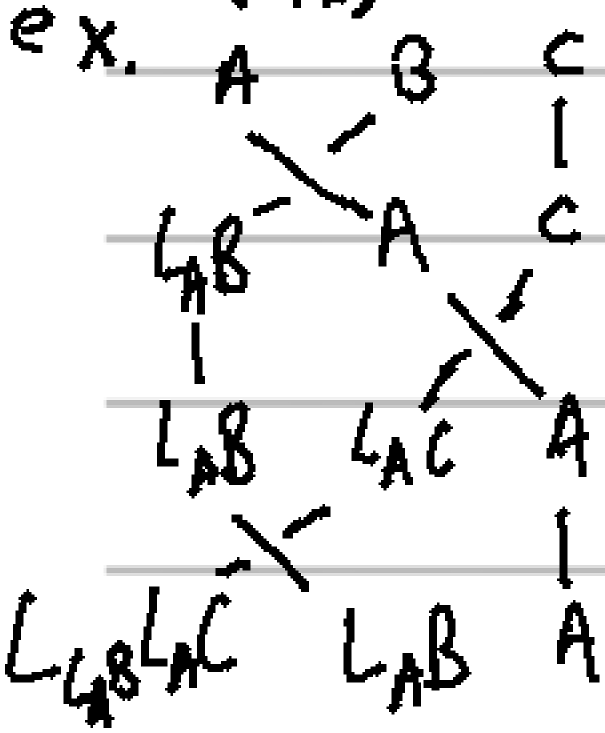
$$(A,B) \longrightarrow (L_A B, A)$$

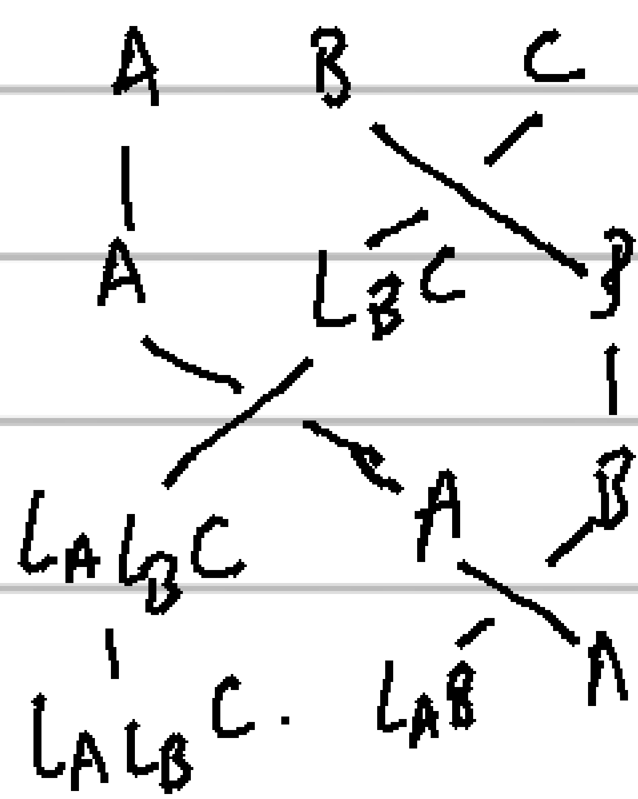
i.e. a mutation takes an exceptional collection to an exceptional collection.

i.e. if σ_i is left mutation by i^{th} object, σ_i^{-1} exists.

(A,B,C)

ex.





#4 is simply the braid relation!

(NB: we only care about isomorphism classes of exceptional collections here, get a braid group action on these classes).

$(\mathcal{O}(-1)[1], \mathcal{O})$ $\qquad\qquad$ $(\mathcal{O}, \mathcal{O}(1))$

$\qquad\|$

$(L_{\mathcal{O}}(\mathcal{O}(1)), \mathcal{O})$

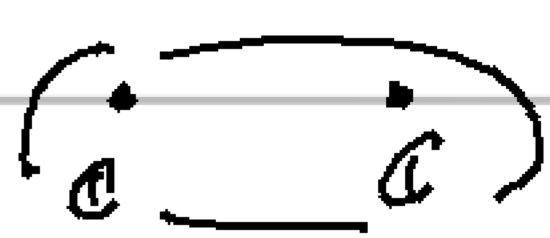
$\mathcal{O}(-1) \to \mathcal{O} \oplus \mathcal{O} \xrightarrow{\binom{x}{y}} \mathcal{O}(1)$, i.e.

$\mathrm{Cone}\{\mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(1)\} \cong \mathcal{O}(-1)[1]$

Not all exceptional collections well behaved!

eg.

$(\mathcal{O}(-1)[1], \mathcal{O}(-n))$ for $n > 0$



, no way to get to full derived category!

full exceptional collection, but extension category doesn't generate.

<u>Def'n</u>: A directed A_∞ category is a strictly unital A_∞ category with a finite ordered set of objects $\{Y_i\}$ such that

$$\mathrm{Hom}_{\vec{A}(Y_i)}(Y_i, Y_j) = \begin{cases} 0 & \text{if } i > j \\ \mathbb{K}e_{Y_i} & \text{if } i = j \\ \text{fin. dim. over } \mathbb{K} & \text{for } i < j \end{cases}$$

denote $\vec{A}(Y_i)$

In $H(\vec{A}(Y_i))$, this is an exceptional collection.

universal property: F A_∞ functor,

$$F : \vec{A}(Y_i) \longrightarrow T$$

T triangulated means there's automatically a

$$\tilde{F} : Tw\,\vec{A}(Y_i) \to T.$$

In $Tw\,\vec{A}(Y_i)$, can define mutations.

$$\{Y_i\} \subset Tw\,\vec{A}(Y_i)$$

mutate

full subcat $\{Z_i\}$

i.e. $Tw\,\vec{A}(Y_i) \simeq Tw\,\vec{A}(Z_i)$

(On the level of K groups, you're changing basis).

Set of directed A_∞ categories

$$A^{\rightarrow}(Y_i) \xrightarrow{\text{mutations}} A^{\rightarrow}(Z_i)$$

gives us a braid action on