Day 4 Talk 2 : Nick R.,

Lefschetz Fibrations

Let (M, ω) be a symplectic manifold.

Def: A Lefschetz fibration is a map $M \to D \ (= D^2)$ s.t.

- crit f are isolated, distinct values
- the fibers of f are symplectic m'folds and

around each crit. pt. $p \in M$,

$\exists$ charts $\phi : \underset{\ni p}{U} \to \mathbb{C}^n$,

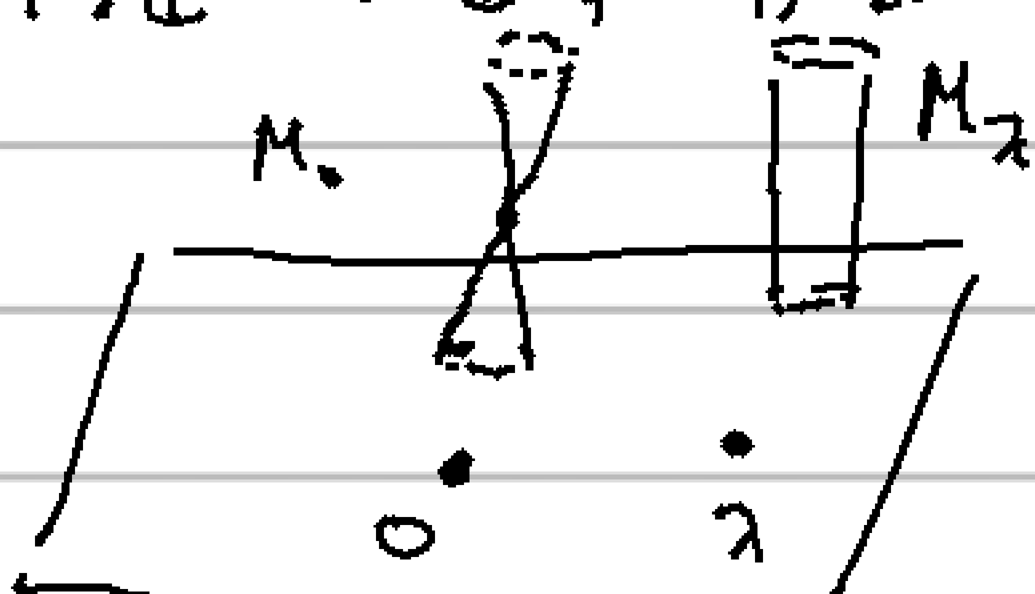
$\psi : V^{\ni f(p)}$ s.t. in these charts, f is given by

$$z_1, \ldots, z_n \longmapsto \sum z_i^2$$

further technical condition which makes these charts interact nicely w/ ω.

Example: Local model

$$f : \mathbb{C}^2 \to \mathbb{C}, \quad (z_1, z_2) \mapsto z_1^2 + z_2^2$$



Symplectic Parallel Transport

Suppose $\gamma : [0,1] \to D^2$ is a path avoiding critical values.

Want: symplectomorphism $\rho_\gamma : M_{\gamma(0)} \to M_{\gamma(1)}$.

Take $p \in f^{-1}(\gamma(0))$

$$T_pM = T_pM_{\gamma(0)} \oplus \underbrace{\mathbb{C}}_{(T_pM_{\gamma(0)})^{\perp,\omega}}$$

We get a canonical lift of the vector field $\frac{\partial}{\partial t}$ along γ.

ρ_γ is the flow along this vector field.

Let $\gamma : [0,1] \to D$ be a path from a basepoint z_0 to a critical value

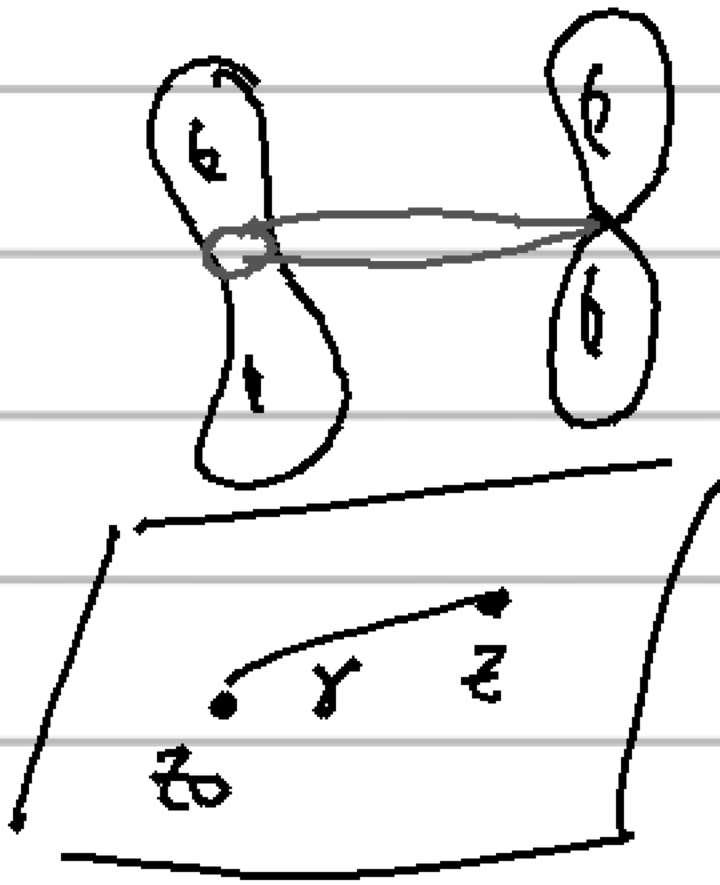
"vanishing path."

Consider the following subset

$$\Delta_\gamma := \{ y \in M_{\gamma(s)} ;\ 0 \le s < 1,\ \lim_{t \to 1} \rho_{\gamma|_{[s,t]}}(y) = x \} \cup \{x\}$$

x is the unique critical pt. in $M_{\gamma(1)}$

In the local model, $(z_i) \mapsto \sum z_i^2$, this corresponds to setting the $z_i \in \mathbb{R}$



Let $V_\gamma = \partial \Delta_\gamma \subset M_{\gamma(0)}$

It is a Lagrangian sphere.

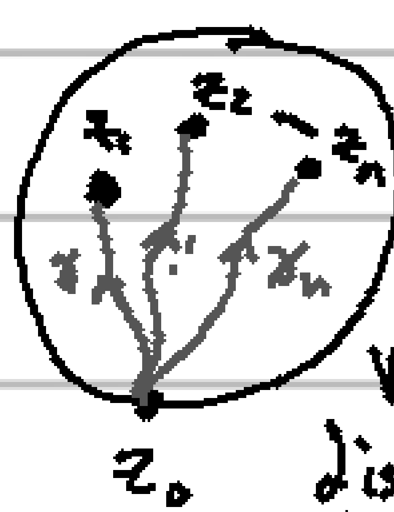
Let $f: M \to D$ be a Lefschetz fibration. Let M_0 be the fiber at a fixed basepoint z_0.

Think of z_0 as lying on ∂D.

Choose a path γ_i from z_0 to each critical value z_i.

We have a collection of paths $\gamma_1, -, \gamma_m$ from z_0 to $z_1, -, z_m$.

Such a collection is called <u>admissible</u> if these paths are ordered by the tangent directions at z_0.



For an admissible collection of paths, the corresponding vanishing cycles $V_1, -, V_m$ are called a <u>distinguished</u> <u>basis</u> of vanishing cycles.

In this situation, we can define the directed Fukaya category $\overrightarrow{Lag}_{\{V_i\}}(M,f)$ using the collection $\{V_1, \dots, V_n\}$

i.e. $\mathrm{Hom}(V_i, V_j) = \begin{cases} 0 & i > j \\ \mathbb{K} e_j & i = j \\ CF^*_{M_0}(V_i, V_j) & i < j \end{cases}$

Let $\mathcal{F}(M,f) = \mathrm{Tw}\,\overrightarrow{Lag}_{\{V_i\}}(M,f)$ be the triangulated envelope of $\overrightarrow{Lag}_{\{V_i\}}(M,f)$.

Thm: $\mathcal{F}(M,f)$ "The Fukaya category of the Lefschetz fibration" is an invariant of the fibration.

(NB: this is called $\mathcal{FS}(M,f)$ in the literature, I believe).

Note: $\{V_i\}$ form a full exceptional collection for $\mathcal{F}(M,f)$.

Want to consider: isotopy classes of paths from z_0 to each z_i.

Consider $\mathcal{D} \subset \mathrm{Diff}^+(D)$ which fix z_0 and map $\{z_1, \dots, z_m\} \to \{z_1, \dots, z_m\}$.

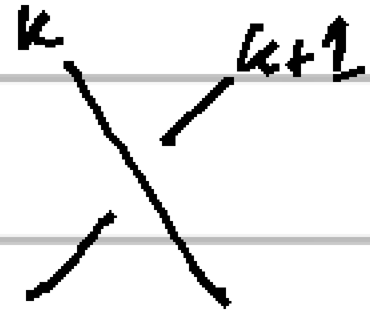
This group acts transitively on isotopy classes of paths. Need to consider $\pi_0 \mathcal{D}$

<u>Thm</u>: $\pi_0 \mathcal{D} = Br_m$

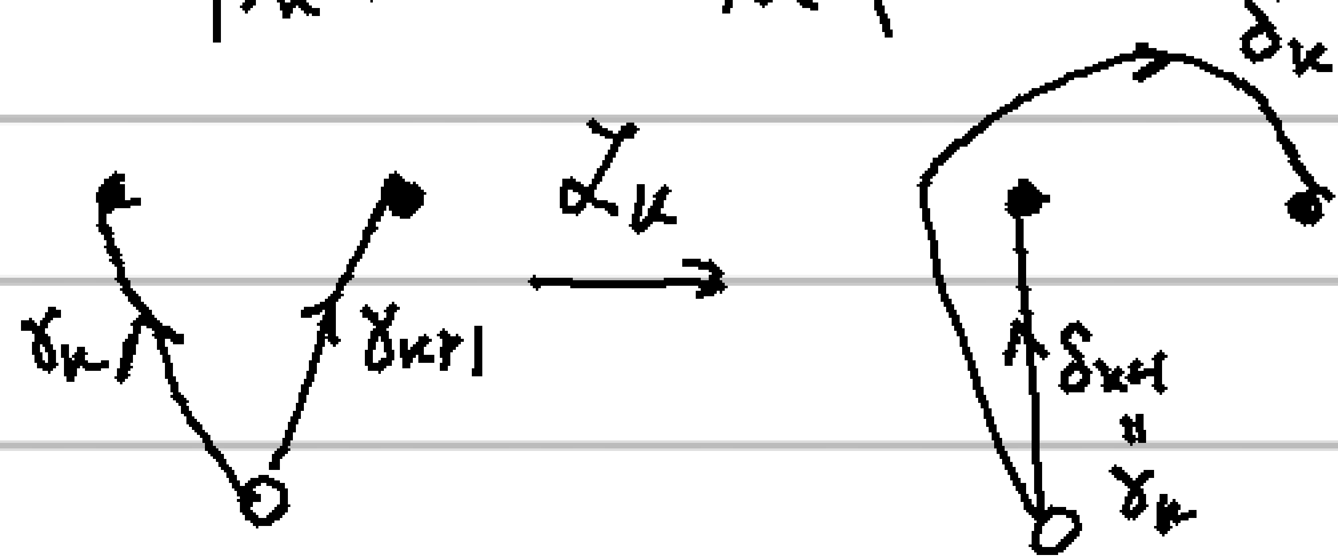
$\Rightarrow$ Br_m acts simply and transitively on isotopy classes of paths.

Br_m is generated by elements

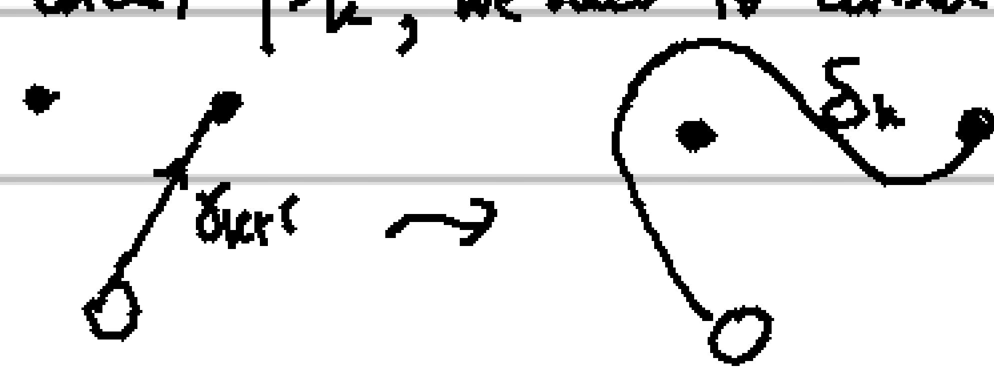
β_k :



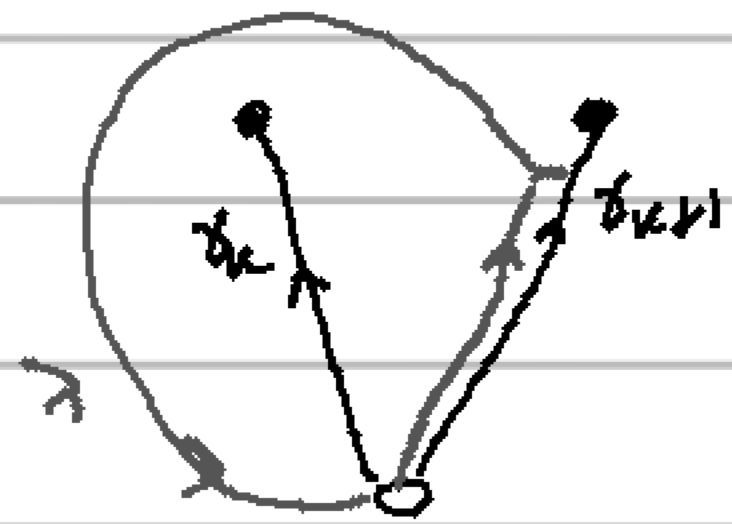
How does β_k act on the paths?



i.e. to see what happens to our distinguished basis under β_k, we need to consider



Compare $V_{\gamma_{k+1}}$ to V_{δ_k}:



<u>Upshot</u>: The difference between V_{δ_k} will be the monodromy around z_k.

<u>Thm</u>: (Symplectic Picard-Lefschetz theorem):
Let $f: M \to D$ be a Lefschetz fibration, γ a vanishing path, λ a loop in $D - \mathrm{crit} f$ which doubles γ, winding counterclockwise around $\gamma(1)$.
<u>Then</u>: the monodromy around λ is (Ham.) isotopic to a (symplectic) <u>Dehn twist</u>* along V_γ, i.e.

$$\mathrm{mon}_\lambda = \tau_{V_\gamma}$$

<u>Dehn Twists</u>: M symplectic mfold, $V \subset M$ Lag'n sphere
Dehn twist $\tau_V \in \mathrm{Aut}(M, \partial M)$
τ_V will be supported on a nbhd of V

$\Rightarrow$ Suffices to define τ_V when $M = T^*V$.

When $V = S^1$

T^*V :

<u>Thm:</u> [Seidel] Let V be a Lagrangian sphere in M, L a Lagrangian in M.

Then,

$$\tau_V(L) = \overbrace{T_V(L)}^{\text{mutation.}}$$

in the Fukaya category.

<u>Cor:</u> $\{V_1, \dots, V_m\} \longrightarrow \{V_1, \dots, V_{\delta_k}, V_{\delta_k}, V_{k+2}, \dots, V_m\}$ are related by mutation, so by Parker's talk, get same derived category.

(Dehn twist is an $SO(n+1)$ equivariant aut. on T^*S^n)