Day 4 Talk 3: Bohan,

HMS for Fanos

1. HMS (for toric Fanos)
2. $D^b(\mathbb{P}^n)$
3. LG mirror of $\mathbb{P}^n$ & FS.

(Paul: these directed Fuk categories were introduced by Kontsevich)

HMS

$$X \overset{\text{mirror}}{\rightsquigarrow} Y \qquad \text{Calabi-Yau}$$

$$DFuk(X) \cong D^bCoh(Y)$$

HMS for toric Fanos:

$$X \ (\dim = n) \rightsquigarrow (\mathbb{C}^*)^n, \quad W: (\mathbb{C}^*)^n \to \mathbb{C}$$

LG superpotential

Ex: $X = \mathbb{P}^2$

$\mathbb{P}^2$ $\quad M = (L, \nabla) \rightsquigarrow$

D anticanonical divisor $\quad |z_1|, |z_2|$

moment map

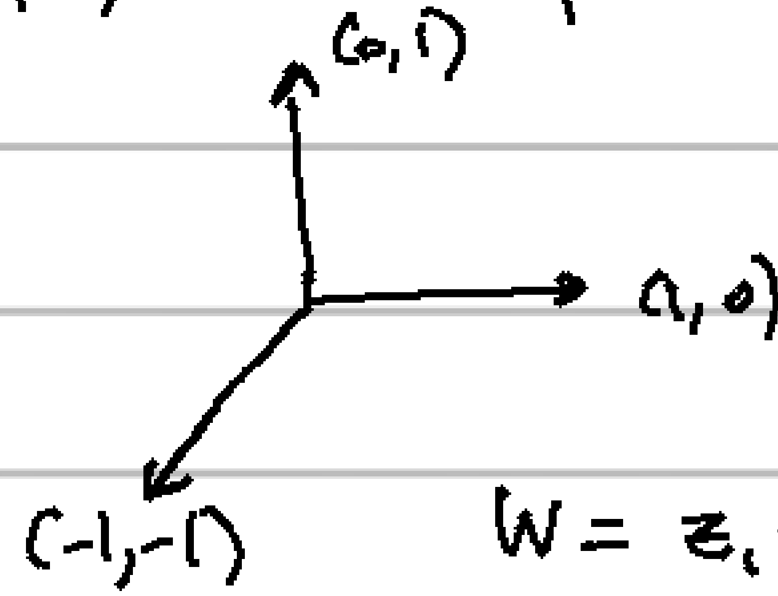
$$M = (\mathbb{C}^*)^2 = \{(\xi_1, \xi_2)\}$$

$$(|z_1| \cdot e^{hol_1(\nabla)}, |z_2| \cdot e^{hol_2(\nabla)})$$

$W = m_2(L, \nabla)$

$= \sum_{\beta} n(\beta) e^{\int_\beta \omega} \cdot hol_{\partial\beta}(\nabla)$.

Fan for $\mathbb{P}^2$:



$W = z_1 + z_2 + \frac{1}{z_1 z_2}$

($\Sigma(1) = \{v_1, \dots, v_n\}$ $\quad W = \sum z^{v_i}$).

1-dim'l part of fan

(Cho-Oh)

collection of vectors spanning the fan.

Takeaway message if you don't know toric geometry: For $\mathbb{P}^2$, the mirror is

$(\mathbb{C}^*)^2 \xrightarrow{W} \mathbb{C}$, $W = z_1 + z_2 + \frac{1}{z_1 z_2}$

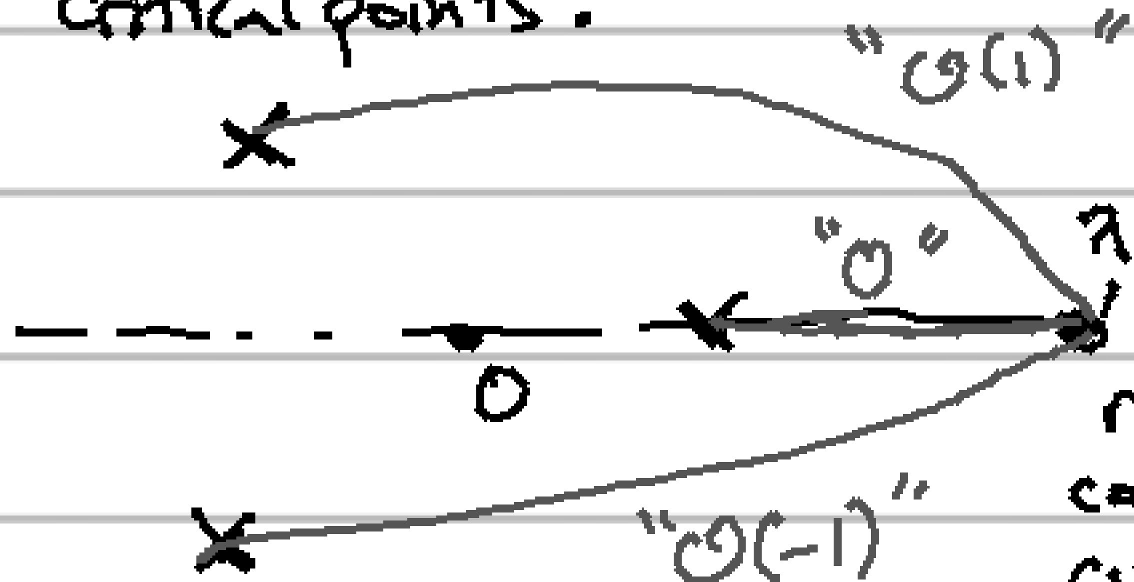
HMS: $D^b(X) \cong D\vec{\mathcal{F}}((\mathbb{C}^*)^n, W)$.

LG($\mathbb{P}^2$)

$$(\mathbb{C}^*)^2 \xrightarrow{W} \mathbb{C}$$

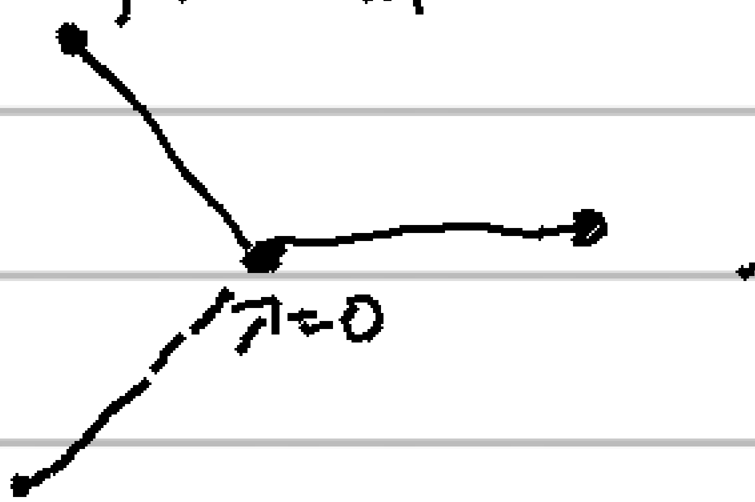
$$W = z_1 + z_2 + \frac{1}{z_1 z_2}.$$

Three critical points:



ref fiber construct vanishing cycles in $W^{-1}(\lambda)$.

Auroux-Katzarkov-Orlov: pick $\lambda = 0$, look at



2. $D^b Coh(\mathbb{P}^n)$:

- $\mathcal{A}$ abelian cat.
- $C^b(\mathcal{A})$ chain complexes
- $K^b(\mathcal{A})$ homotopy
- localize w.r.t. q-isos.

$D^b Coh(\mathbb{P}^n)$ has a full strong exceptional collection.

$D^b(\mathbb{P}^1)$ $\quad \mathcal{O}, \ \mathcal{O}(1)$

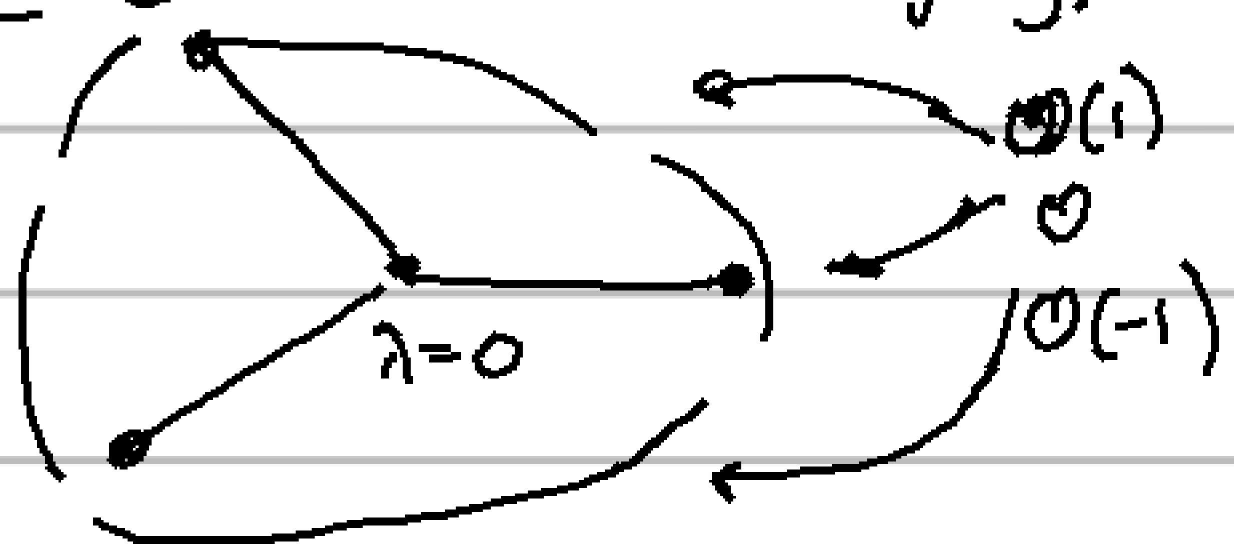
$D^b(\mathbb{P}^n)$ $\quad \mathcal{O}(k), \ldots, \mathcal{O}(k+n)$

$$Hom(\mathcal{O}(i), \mathcal{O}(j)[k]) = 0 \quad i<j$$

strong: every non-zero hom happens in degree zero.

(Paul: historic relic. look at underlying A_∞ category, & mod it down via HPT so all morphisms are degree zero, then strong $\Rightarrow$ no higher products, so historically people used these.).

Full: Generates whole category.



$\mathbb{P}^2$: $\quad \mathcal{O}(-1) \quad \mathcal{O} \quad \mathcal{O}(1)$

$\qquad \mathcal{O}(-1) \quad L_{\mathcal{O}}\mathcal{O}(1) \quad \mathcal{O}$

$\rightsquigarrow$ by def'n

$$\begin{tikzcd} L_0\mathcal{O}(1) \arrow[r] & \mathrm{Hom}(\mathcal{O},\mathcal{O}(1))\otimes\mathcal{O} \arrow[d] \arrow[r, squiggly] & \mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O} \\ & \mathcal{O}(1) \arrow[ul] & \end{tikzcd}$$

But we know we have the Euler sequence: (in $\mathrm{Coh}(\mathbb{P}^2)$ as abelian cat.):

$\mathrm{Coh}(\mathbb{P}^2)$:

$$0 \to \Omega^\bullet \to \mathcal{O}(-1)\oplus\mathcal{O}(-1)\oplus\mathcal{O}(-1) \to \mathcal{O} \to 0$$

$$0 \to \Omega^\bullet\otimes\mathcal{O}(-1) \to \mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O} \to \mathcal{O}(1) \to 0$$

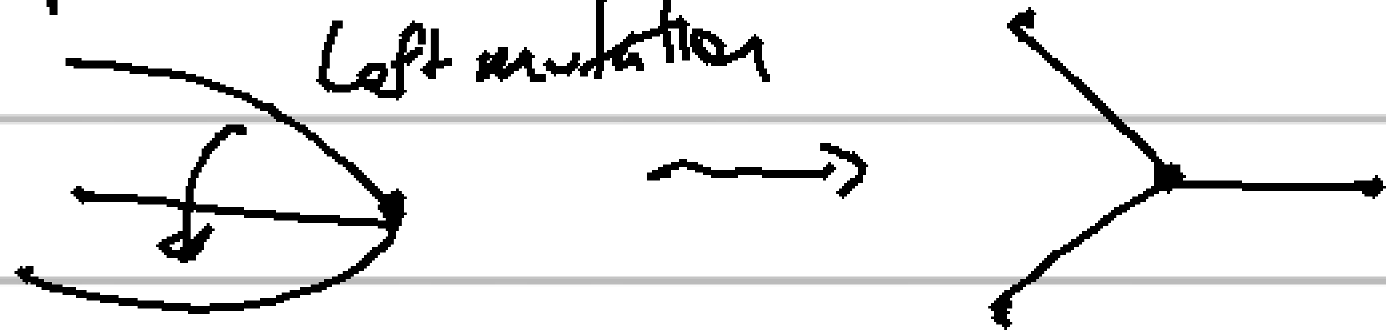
$$\wr\, D^b$$

$$\to \Omega^\bullet\otimes\mathcal{O}(1) \to \mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O} \to \mathcal{O}(1) \to$$

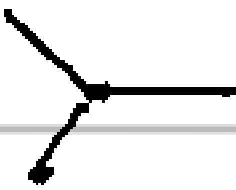
$$\begin{tikzcd} H_0 \arrow[d, squiggly] & H_1 \arrow[d, squiggly] & H_2 \arrow[d, squiggly] \\ \mathcal{O}(-1) \arrow[r, shift left=2, "x_0"] \arrow[r, "y_0"] \arrow[r, shift right=2, "z_0"'] \arrow[rr, bend right=30] \arrow[rr, bend right=40] \arrow[rr, bend right=50] & \Omega^\bullet(1) \arrow[r, shift left=2, "x_1"] \arrow[r, "y_1"] \arrow[r, shift right=2, "z_1"'] & \mathcal{O} \end{tikzcd}$$

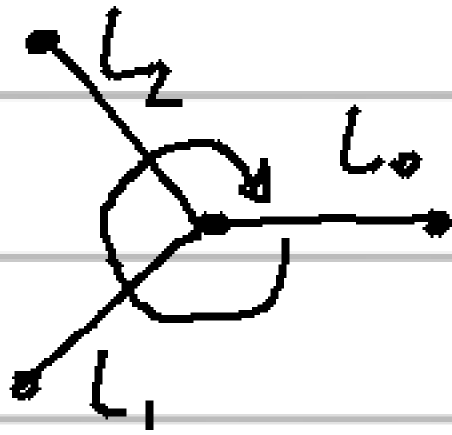
Result: $\mathrm{Hom}(H_i, H_j) \cong \Lambda^{j-i}$

On symplectic side:



Let's do the computation!

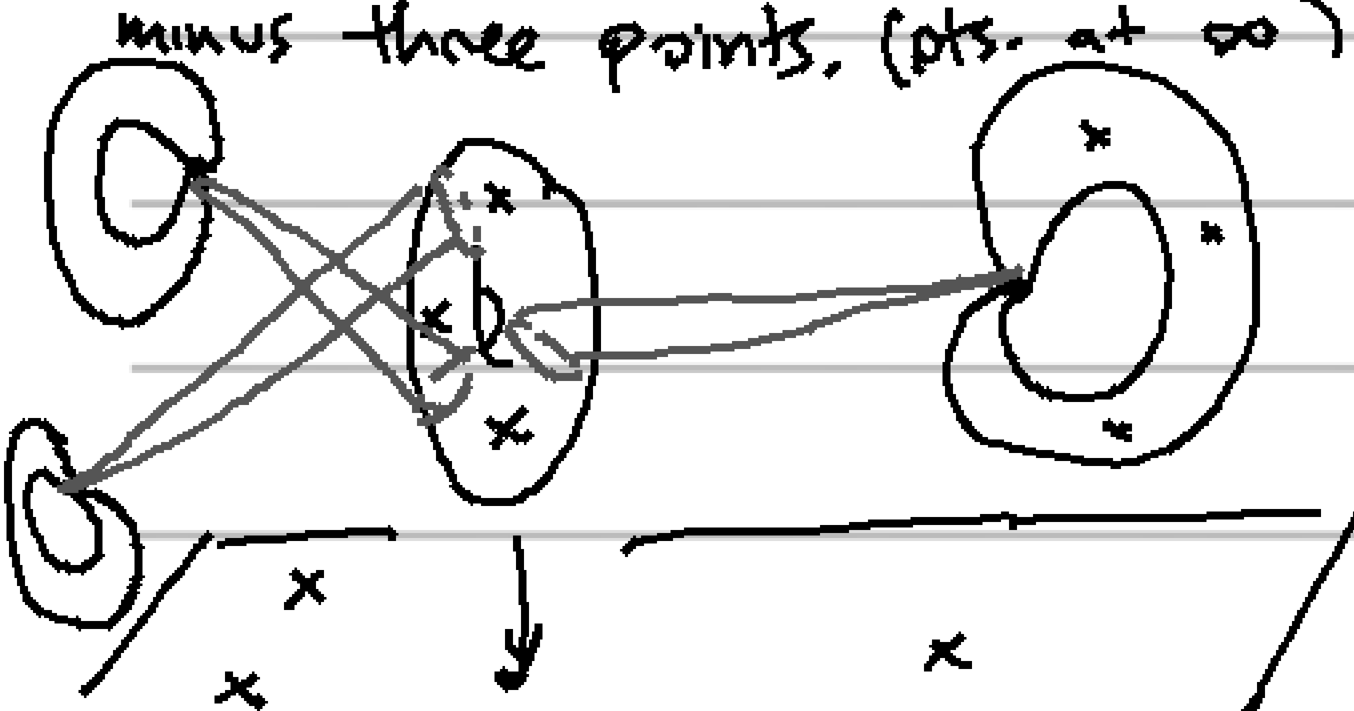
3. $\mathcal{F}^{\to}((\mathbb{C}^*)^2, W,$  $)$



$$\Sigma_\lambda = W^{-1}(\lambda)$$

$$\rightsquigarrow z_1 + z_2 + \frac{1}{z_1 z_2} = \lambda.$$

When this is not singular, it is an elliptic curve minus three points. (pts. at ∞)

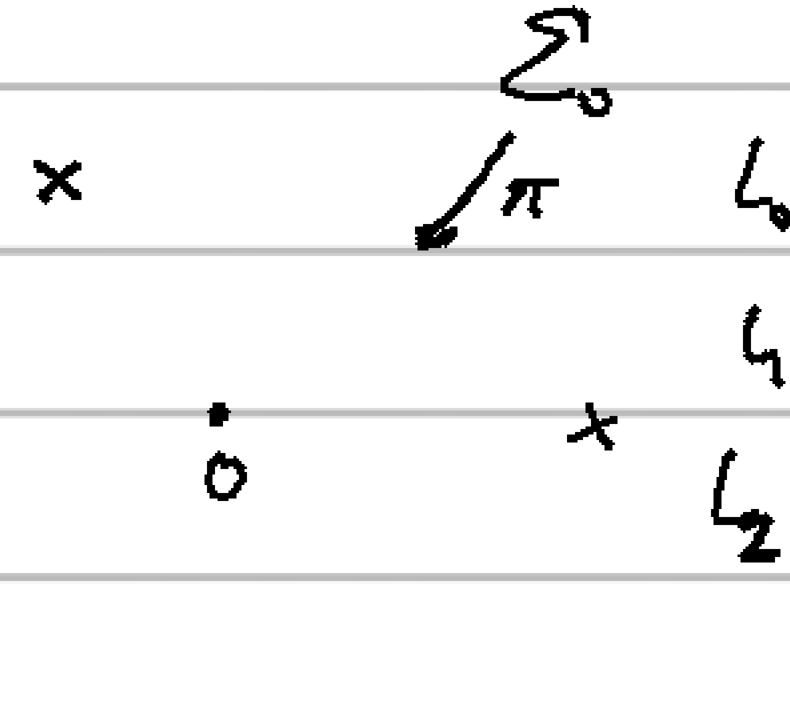


$\pi_\lambda : \Sigma_\lambda \longrightarrow \mathbb{C} \qquad (z_1, z_2) \longmapsto$ to

$\lambda_0 = 0 \quad \pi_0 : \Sigma_0 \longrightarrow \mathbb{C} \quad (z_1, z_2) \longmapsto z_1.$

$$\begin{array}{c} \Sigma_0 \\ \big\downarrow \pi \\ \mathbb{C} \end{array}$$

branched covering



$\pi_\lambda : \Sigma_\lambda \longrightarrow \mathbb{C}$

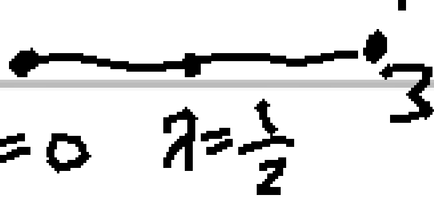
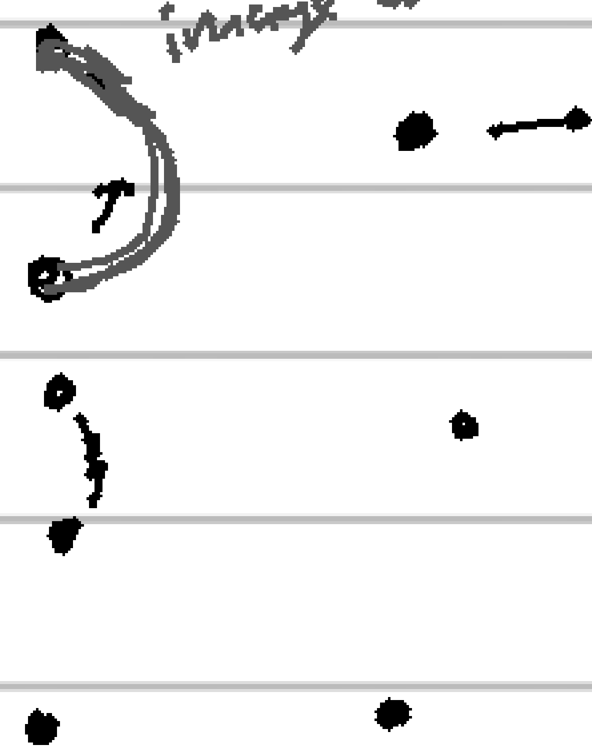


image of vanishing cycle of Σ_0

$\Sigma_0 \longrightarrow \mathbb{C}$

$\Sigma_{1/2} \longrightarrow \mathbb{C}$

$\Sigma_3 \longrightarrow \mathbb{C}$



2 of the branch points come together.

Σ_0

$\downarrow \pi$

$\pi^{-1}(\delta_0) = L_0'$.

<u>Claim</u>: $L_0' \rightsquigarrow L_0$ (topological vanishing cycle).

Since dimension = 2.

L_0' — inv. under complex conjugation (by picture above)

L_0 — inv. under complex conjugation, b/c std. ω on $(\mathbb{C}^*)^2$

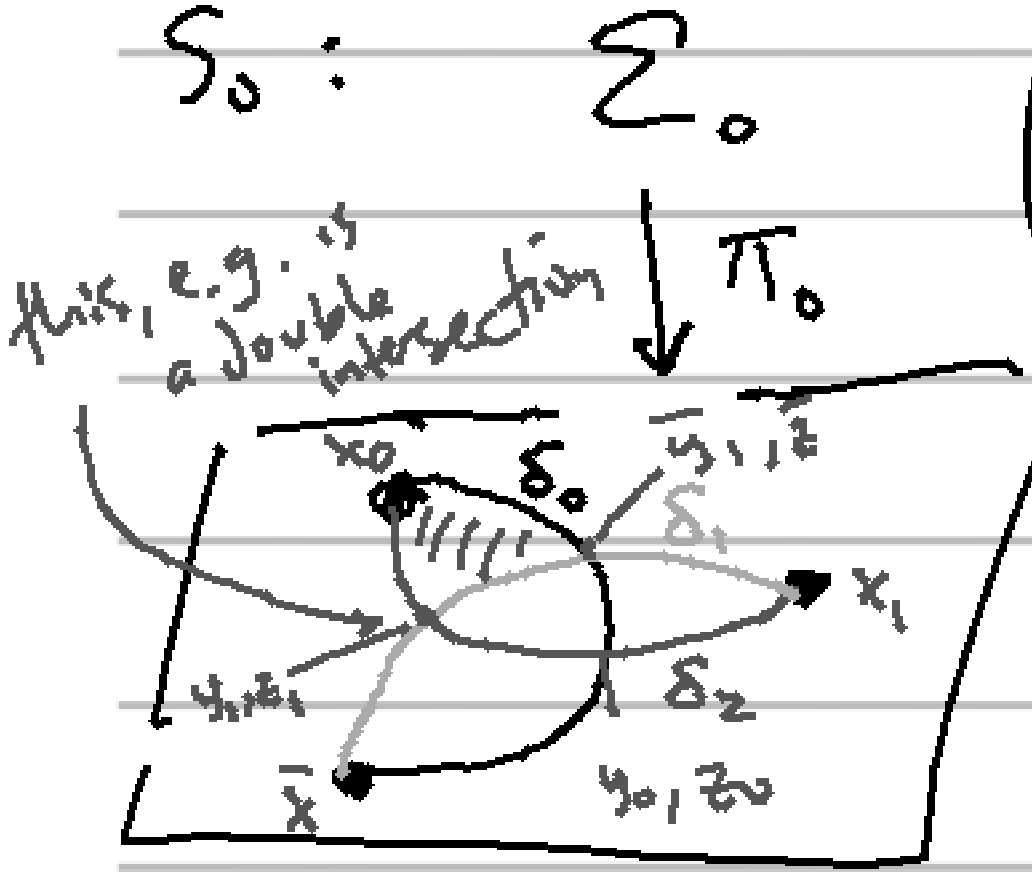
$$= \omega = \frac{1}{z_1} dz_1 \wedge \frac{1}{\bar{z}_1}$$

$$J\hat{\omega} = -\omega.$$

$\Rightarrow L_0' \underset{\text{Ham. isotopy.}}{\sim} L_0$

So, can replace L_0 by L_0'.

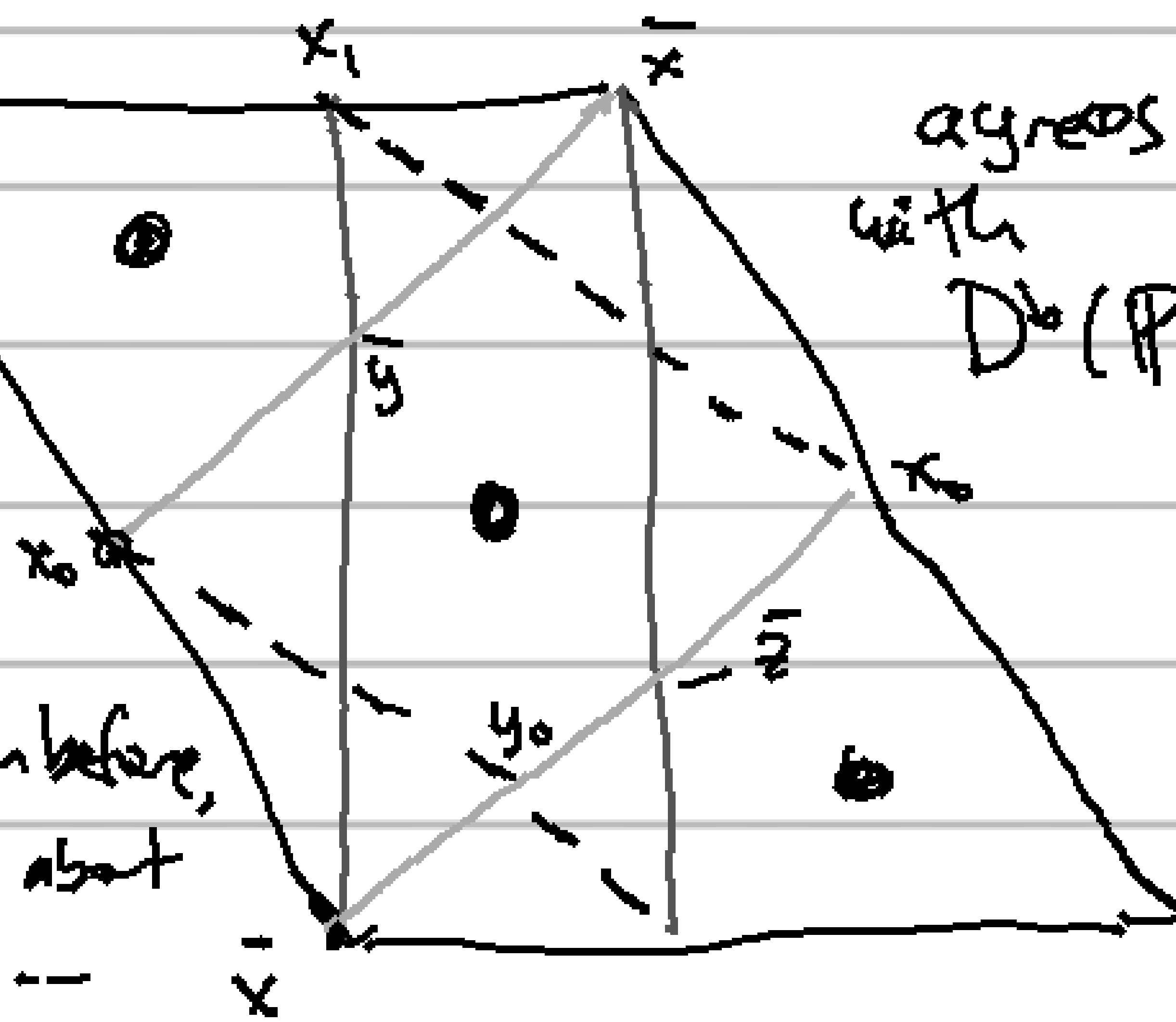
S_0 :



(Recap: Signed areas bound by L_0, L_0' are the same, namely 0, so b/c this is dimension 2, they are Ham. isotopic.)

$(L_0', L_1', L_2') \xrightarrow{\pi_0} (\delta_0, \delta_1, \delta_2)$.

Final picture:



agrees with $D^b(\mathbb{P}^2)$.

$\hom(L_i', L_j')$
$\|$
Λ^{j-i} from before,

make an argument about higher products, ...

♡ Questions:

John: $\mathcal{F}(X)$ seems to involve some Λ, but $D^b(\check{X})$ doesn't. What's the deal?

Paul: Actually, if you do $D^b(\check{X})$ right, it's a smooth proper scheme over Λ.

Scott: Is there are a similar statement when $\mathcal{F}(X)$ has only $\mathbb{Z}/2$-coeffs?

Interesting questions -- arithmetic aspects ~~are~~ lead to interesting scenarios sometimes.

Mirror symmetry does apply for $\mathbb{Z}/2$ graded things.