Day 4 Talk 4: Discussion

Possible topics:

1. Twisted complexes and what does "derived" mean?
2. Signs (of madness)
3. Grading on HF^*.
4. open-closed string theory
5. obstructions & curvature
6. Koszul duality and mutations. , has to do w/ Lefschetz fibrations
7. immersed Lagrangians

Curved A_∞-structures:

Ob $\mathcal{A}$

$\hom_{\mathcal{A}}(X, Y)$

$$\mu^d_{\mathcal{A}} : \hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes \hom(X_0, X_1) \to \hom_{\mathcal{A}}(X_0, X_d)[2-d] \quad (d \geq 0).$$

Bianchi

$$\mu^1_{\mathcal{A}}(\mu^0_{\mathcal{A}}) = 0 \in \hom^3(X, X)$$

$$\mu^1_{\mathcal{A}}(\mu^1_{\mathcal{A}}(x)) = \mu^2_{\mathcal{A}}(\mu^0_{\mathcal{A}}, x) \pm \mu^2_{\mathcal{A}}(x, \mu^0_{\mathcal{A}})$$

Ex: $E \to M$ vector bundle w/ connection

$\mathcal{A} = \Omega^*(M, \mathrm{End}(E))$

$\mu^1 = d_\nabla \quad \mu^2 = \wedge, \quad \mu^0 = F_\nabla$

Ex 2: $E \to M$ v.b. over a complex mfld

$$\mathcal{A} = \Omega^{0,*}(M, \mathrm{End}(E))$$

$$\mu^1 = \bar\partial_\nabla, \quad \mu^0 = F_\nabla^{0,2}.$$

$d^2 = 0$ has interesting solutions (thanks to structure of nilpotent matrices)

$d^2 = 1$ less interesting solutions (get a splitting).

($\mu^0 = 0 \longleftrightarrow$ vector field vanishes at 0 (thinking about these as curvatures)

But maybe the v.f vanishes somewhere else! try to change/translate (infinitesimally) to extract something meaningful.)

Technical condition: $\mathcal{A}$ defined over $\mathbb{Z}[[t]]$, μ^0 of order t. (use formal deformation theory to make it go away).

$\mathcal{A}$ (obstructed) $\Rightarrow \tilde{\mathcal{A}}$ (unobstructed) (similar to twisted complexes).

$$\mathrm{Ob}(\tilde{\mathcal{A}}) = \{(X, \alpha) \mid X \in \mathrm{Ob}\,\mathcal{A},\ \alpha \in \hom^2_{\mathcal{A}}(X, X),$$

of order t

$$\underbrace{\mu^0 + \mu^1(\alpha) + \mu^2(\alpha, \alpha) + \cdots}_{-\mu^0_{\tilde{\mathcal{A}}}} = 0$$

inhomogeneous Maurer-Cartan

Alternatively, just consider category of all such objects, & throw away obstructed guys.

(FOOO call this "filtered" A_∞ alg.)

You can do this & twisted complexes at once, but not related.

Version of this you can do which hasn't really been written down.

If connection is projectively flat, induced connection on End bundle is flat.

(think of HH as being central).

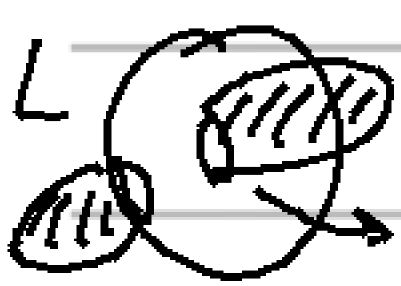
$\rho \in CC^2(A, A)$
$d\rho = 0$. of order t

consider $\bar{A}_{[\rho]}$ (not curved)

$Ob(\bar{A}_{[\rho]}) = \{(X, \alpha) \mid \text{---} \ \mu^0 + \mu^1(\alpha) + \mu^2(\alpha, \alpha) + \dots = \rho^0\}$.

Why is this important?

(M, ω) $c_1(M) = 0$



L

small, b/c no hol. disks of area 0 (t power controlled by $\omega(\beta)$).

μ^0 = boundaries of holo. discs

Try to solve

$$\mu^0 + \mu^1(\vec{\alpha}) + \cdots$$

gradually, step by step.

Ex: $L^3 \subset M^6$, $c_1(M) = 0$

$H^1(L) = H^2(L) = 0$

$\Rightarrow \exists$ unique object in $\mathcal{F}(\cdots)$

(i.e. can solve, & solve uniquely)

For other cases, generally an infin.-dim'l space of paths/choices

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physics: Objects $(L, A \in \Omega^1(L, \mathbb{R}))$,

$\omega|_L = 0$, $F_A = 0$ (to guarantee good boundary conditions in twisted σ-model).

this is before instanton corrections.

Actually, what's true is $\frac{i}{2\pi} F_A = PD(\mu^0)$.

form w/ singularities. current

Max's examples

$\rho = \text{const.} \cdot \text{id}$, so can translate away.

($c_1(M) \neq 0$, so not graded any more)

(7) Immersed Lag'ns (Joyce, -)

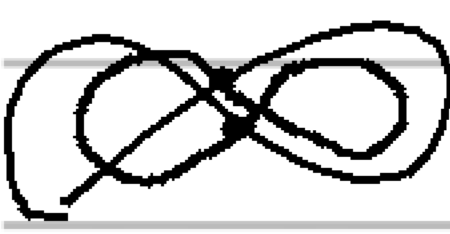
$L \xrightarrow{\iota} M$ Lagrangian immersions

(lots of examples & existence thms).

$$CF^*(L,L) = C^*(L) \oplus \bigoplus_{\substack{(x,y)\in L \\ \iota(x)=\iota(y) \\ x\neq y}} \mathbb{Z}\langle x,y\rangle$$

Morse complex

(why ordered pairs?



when you flow, 2 isection points!)

μ^0 = holo discs +

claim: everything else is the same.

i.e. $u : \mathbb{H} \xrightarrow{J\text{-hol.}} M$ $\qquad \lim_{s\to+\infty} \tilde{u}(s) = x$

$\tilde{u} : \mathbb{R} \longrightarrow L$ $\qquad \lim_{s\to-\infty} \tilde{u}(s) = y$.

$\iota(\tilde{u}) = u|_{\mathbb{R}}$

ex:  seems unobstructed to naked eye, but look at differential.

Morally, think of all solutions to Maurer-Cartan as arising from isotopies?

Signs and gradings

$L_0, L_1 \subset M$

$\mathcal{P} = \{ x : [0,1] \to M \mid u(0) \in L_0,\ u(1) \in L_1 \}$

$\mathcal{P} \longrightarrow U_\infty / O_\infty$

① $H^1(U_\infty/O_\infty) = \mathbb{Z} \longrightarrow H^1(\mathcal{P})$ obstruction to grading

② $H^2(U_\infty/O_\infty; \mathbb{Z}/2) = \mathbb{Z}/2 \longrightarrow H^2(\mathcal{P}; \mathbb{Z}/2)$ obstruction to signs.

① $2c_1^{rel} \in H^2(M \times [0,1], L_0 \times \{0\} \cup L_1 \times \{1\})$

$\big\downarrow$ transgression

$H^1(\mathcal{P})$

try to kill this

② $w_2(L_0) \oplus w_2(L_1) \in H^2(L_0) \oplus H^2(L_1)$

$\big\downarrow$ evaluation

$H^2(\mathcal{P}; \mathbb{Z}/2)$

(some version of family Atiyah-Singer index thm.)

① some sort of Calabi-Yau condition, but more.

Killing $2c_1$:

(M,ω,J) fix $(\Lambda^n_{\mathbb{C}} TM)^{\otimes 2} \cong \underline{\mathbb{C}}$

("you don't want to just know it's dead, you want to kill it explicitly")

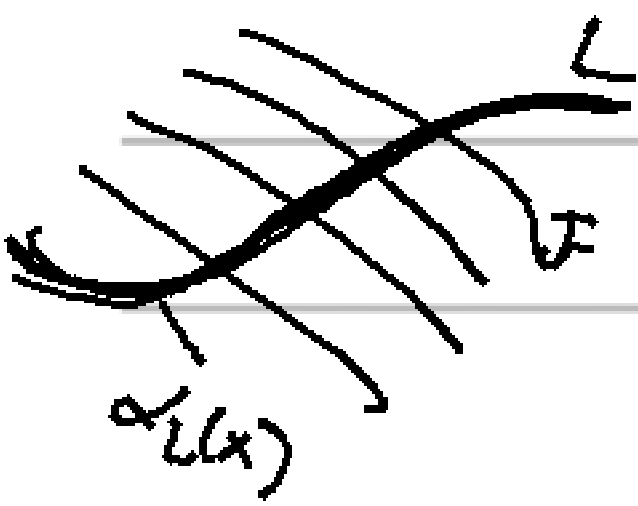
Let's say we're killing c_1: choose complex n-form $\eta \in (\Lambda^n_{\mathbb{C}} TM)$.

$L^{\text{oriented}} \subset M \longrightarrow \mu_L \in H^1(L)$

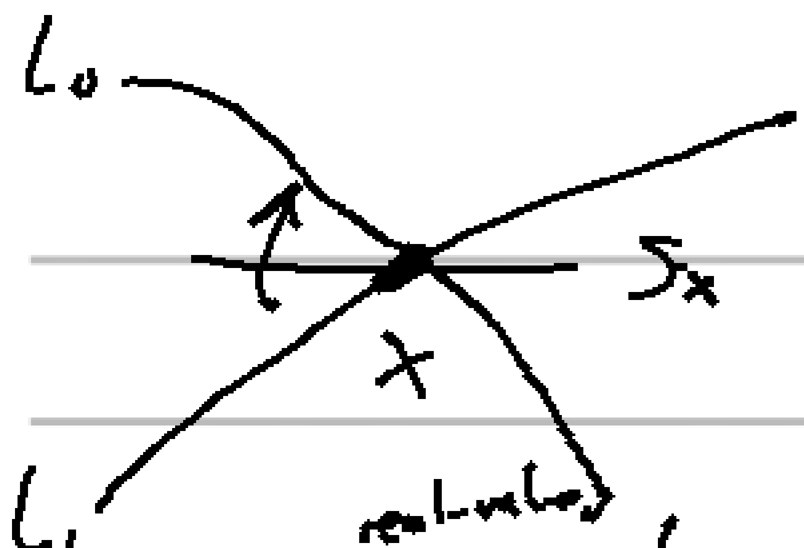
$\mu_L = [\alpha_L : L \longrightarrow S^1]$, $\alpha_L(x) = \eta(\underset{\text{orthonormal basis.}}{TL_x})_{\text{phase}}$

Choose a real-valued phase, a lift $\tilde{\alpha} : L \to \mathbb{R}$ of α.

M surface: trivializations of TM = oriented foliation, F



α_L gives you a way of rotating TL into the foliation.



real-valued phase of L_0

$\mathcal{F}_x \rightsquigarrow (TL_0)_x \xrightarrow[\text{way}]{\text{"short"}} (TL_1)_x$ real-valued phase of L_1 $\to \mathcal{F}_x$

count winding #

Thomas-Yau: apply this to uniqueness of special Lagrangian.

This reproduces: the classical grading in $\mathcal{F}_x$ Morse theory.

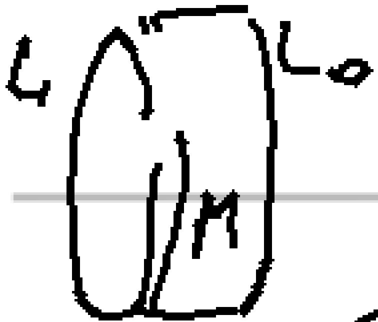
Signs: Pick a spin structure (not an iso class, an actual spin structure)

spin automorphism (-1) acts on HF by (-1) — very messy.

Families index theory

$\mathcal{P} \to U/O$

$\Omega\mathcal{P} \to \Omega(U/O) \simeq \mathbb{Z} \times BO_\infty$ = Fredholm operators



loop in $\mathcal{P}$.

$\rightsquigarrow$ $\bar\partial$-op. on u^*TM with boundary values on L_0, L_1.

$u : [0,1]\times S^1 \to M$.

compute using Atiyah-Singer

<u>Next obstruction:</u>

$$H^2(\mathcal{P};\mathbb{Z}/2) \to H^1(\Omega\mathcal{P};\mathbb{Z}/2) \leftarrow H^1(\mathbb{Z}\times BO_\infty;\mathbb{Z}/2) = \mathbb{Z}/2.$$

determinant line of Fredholm operator

w_1(det line bundle)

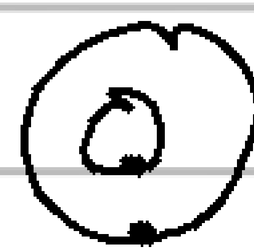
If u varies, & you have a 1-param. family of det-lines, what is the det. line bundle? trivial or not: family index thing. (Atiyah-Singer pt. 5).

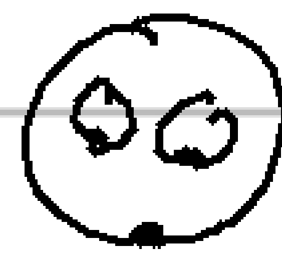
Open-Closed String Theory

How does $F(-)$ fit into an O-C str. theory?

Formal theory:

Open-closed: framed little disc operad.

$\mathcal{O}(1) = \{$  $\} \simeq S^1$

$\mathcal{O}(2) = \{$  $\} \simeq T^3.$

$\mathcal{O}(3) = \text{---}$

operad by shrinking disc & putting in, aligning marked points.

Algebra over $H_*(\mathcal{O})$: BV-algebra

V graded vector space, differential

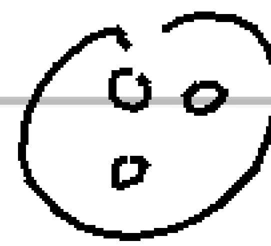
$\Delta : V \longrightarrow V[-1]$

$V \times V \to V$ commutative, associative.

7^3 classes give $x\cdot y$, $x\cdot\Delta y$, $\Delta x\cdot y$, $\Delta(x\cdot y)$;
($\Delta^2 = 0$).

The failure of Δ to be a derivation, is a Lie bracket; i.e.

$$\{x,y\} = \Delta(x\cdot y) - x\cdot\Delta y - \Delta x\cdot y$$

Jacobi from $\mathcal{O}(3) =$  .

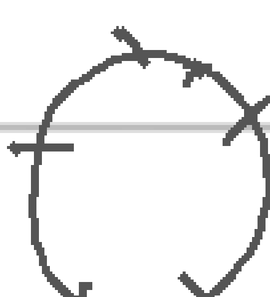
this is a genus 0.
closed string topological field theory (i.e. a BV algebra)

open-closed

$V^{\otimes d} \to V$

$V^{\otimes d} \to W^{\otimes e}$

V, W ← carries a graded symmetric pairing

 $: \mathbb{C} \to W^{\otimes e}$

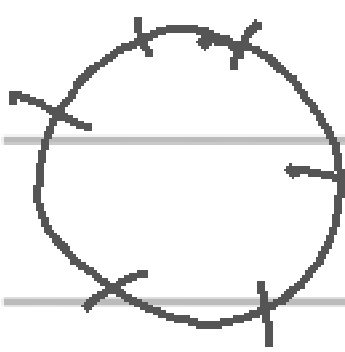
compactification

open-closed top. string theory.

"bi-colored, ($\tilde{\Sigma}$, cyclic) operad"

?

permute boundary pts.



: $C \to W^{\otimes l}$

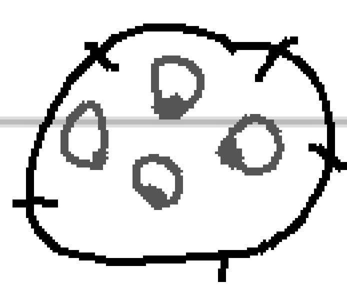
gives W the structure of an A_∞ algebra which is cyclic (C-Y)

(Fuk categories also have this, up to htopy, after lots of work).

homotopy BV-algebra

can look at chains of these guys, but get **no** new info. (some sort of formality result)

mixed guys:



what happens to these?

(D) gives a map of homotopy BV
algebras (similar to Sheel's talk).

(*) $V \longrightarrow CC^*(W,W)$

$HH^*(W,W)$ ↖ (has structure of BV algebra, requires some work).

counts Δ = Connes' operator;
Sheel's product.

should be a formal consequence of product structure of these moduli spaces.

(idea: $C^*(M) = V^{\otimes d} \longrightarrow W^{\otimes d} = CF^*(L,L)$).

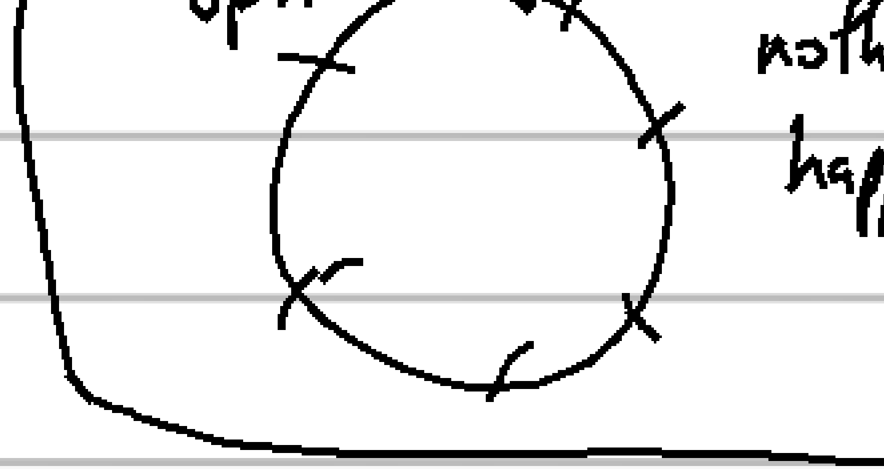
Can recover some closed string stuff from W,
i.e. set $V = CC^*(W,W)$ (*) = id.
(don't require (*) to be a quasi-iso, i.e. $\overline{V}=0$)
but there are classes in which this is the case
(open-closed TFT := alg. over chains on bicolored algebra.)

What actually happens (imprecise)

Moduli spaces are further compactified to Deligne-Mumford spaces

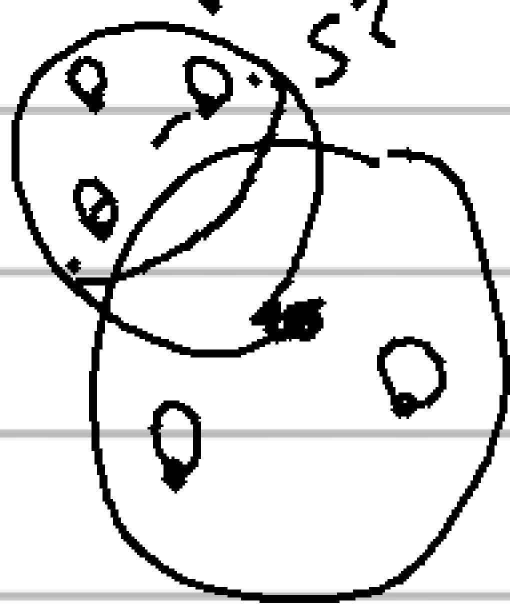
closed

open



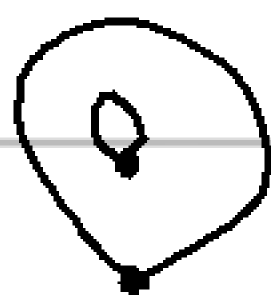
nothing happens

degen



S^2

closed-string Operad $\bar{\mathcal{O}}$.

$\bar{\mathcal{O}}(1) \simeq$  $\cup$  $\simeq pt$

Δ deg

$\bar{\mathcal{O}}(2) =$  $\simeq pt$ - (need to consider all degeneracies)

$\bar{\mathcal{O}}(3) \simeq \bar{M}_{0,4} \simeq S^2$ (Paul's claim). product

genuinely new 3-fold product

open-string: cyclic A_∞-structure. V

closed-string sector: CohFT (cohomological field theory) W

(∞ of generators).

Kontsevich-Manin, Dubrovin, Givental

in general, some relations between product, not enough.

?

open-closed string sector

$V \quad W$

still have $\underset{\text{CohFT}}{V} \to \underset{\text{homotopy BV}}{CC^*(W,W)}$.

If this is a quasi-iso. $\Rightarrow$ Connes boundary operator vanishes

$\Rightarrow$ spectral sequence from $HH^*(W,W)[u]$ to $HC^*_-(W,W)$ degenerates (Connes bd. is first differential).

We would like the spectral sequence to degenerate anyway i.e. get a map of htpy CohFT's.

Kontsevich conjecture: if A_∞ structure "smooth", get this degeneration.

intuition from Kähler geometry; degeneration of Hodge-deRham forms...

On $\mathcal{F}$, we have all these structures, in particular.

$\Delta = 0 \Rightarrow \{-,-\} = 0$ on V.

i.e. $V \to CC^*(W,W)$ map of dgla's bracket is 0.

$\Rightarrow$ family of A_∞-algebras over V (big Fukaya category).

(really useful for, e.g. toric varieties).

(i.e. get a sheaf of A_∞ categories over V, fiber at 0 is usual Fukaya category)

Can, e.g. deform by c_1, which has some geometric significance.

Assoc cat - standard deformation

deg k piece $\rightsquigarrow$ rescale by λ^k.

suppresses higher order term

On the whole, unsatisfactory.

Can't really reconstruct $V^\cdot$ from $CC^*(W,W)$

In principle, known what you have to do to upgrade htpy BV $\rightsquigarrow$ CohFT.

Additional piece of info required (to do w/ killing Δ). First manifestation is a power series, in two variables w/ values in $HC^*_-(W,W)$.

Given W, have different choices of data, give different CohFT's. Group that acts on those is ~~the~~ ∞-dim'l, like a loop group of a symplectic gp "Givental's twisted loop group." cf. Kevin Costello.

Costello knows how to add in the extra data.

There is absolutely a higher genus version of this.