Day 5 Talk 2: James,

Fukaya Cat. of Torus Fibrations

$p : X \to Y$

X symplectic, Y compact base

p locally trivial, Lagrangian torus fibration

Kontsevich–Soibelman 2001:

Relate $\mathcal{F}(X)$ to $\mathcal{O}_Y$-modules,

$\mathcal{O}_Y$ sheaf of (Λ) algebras on Y

Novikov ring.

i.e. $D\mathcal{F}(X) \simeq D^b(\mathrm{coh}\ \mathcal{O}_Y\text{-mod})$

Affine structure on Y

Def: An integral affine structure on Y consists of an atlas $\{\varphi_i : U_i \hookrightarrow \mathbb{R}^n\}$

s.t. $\varphi_i \circ \varphi_j^{-1} \in GL(n, \mathbb{Z}) \ltimes \mathbb{R}^n$

Y integral affine $\longrightarrow p : X \to Y$

$$\begin{array}{c} X = T^*Y / \underbrace{T^*_{\mathbb{Z}} Y}_{\text{local system of integral cotangent vectors.}} \\ \downarrow p \\ Y \end{array}$$

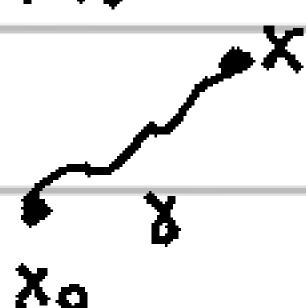
Now, given $\begin{array}{c} X \\ \downarrow p \\ Y \end{array}$ $H_1(\text{fiber}, \mathbb{Z})$ local system on Y. Locally:

take a basis of sections, $\gamma_1, \dots, \gamma_n$

represent them by $(n+1)$ dim'l submanifolds $\Sigma_1, \dots, \Sigma_n \subset X$.

(Paul: these are all Lag'n torus fibrations which <u>admit</u> a Lag'n section).

basepoint x_0.



coords: Define $y_i(x) = \displaystyle\int_{p^{-1}(\gamma) \cap \Sigma_i} \omega$

$$\Lambda := \left\{ \sum_{i=0}^{\infty} c_i T^{\lambda_i} \,\middle|\, c_i \in \mathbb{C}, \lambda_i \in \mathbb{R}, \lim_{i \to \infty} \lambda_i = \infty \right\}$$

Define $\operatorname{val}(\sum c_i T^{\lambda_i}) := \min\{\lambda_i\}$

To construct $\mathcal{O}_Y$, construct $\mathcal{O}_{\mathbb{R}^n}$, along with an action $GL(n, \mathbb{Z}) \ltimes \mathbb{R}^n$ on $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$.

$$\mathcal{O}_{\mathbb{R}^n}(U) = \left\{ \sum_{k=(k_1,\dots,k_n)} a_{k_1 \dots k_n} z_1^{k_1} \cdots z_n^{k_n} \,\middle|\, a_k \in \Lambda, \right.$$

$$\forall y \in U, \lim_{N \to \infty} \inf_{|k| \le N} [\operatorname{val}(a_k) + \langle k, y \rangle] = \infty.$$

(here $k \in \mathbb{Z}^n$, $y \in \mathbb{R}^n$).

How to think about this?

$f \in \mathcal{O}_{\mathbb{R}^n}(U)$ formally a Laurent series over $(\Lambda^*)^n$

Last condition says $f(z_1, \ldots, z_n)$ converges in Λ whenever $\{val(z_1), \ldots, val(z_n)\} \in U$.

Paul: moment when non-Archimedean analytic geometry entered mirror symmetry.

Ex: $\mathcal{O}_{\mathbb{R}}(\mathbb{R}) = \{ \sum_{k \in \mathbb{Z}} a_k z^k \mid a_k \in \Lambda$ as $k \to \pm\infty$, the order (valuation) of a_k should grow more than linearly $\}$

i.e. the analogue of an entire holomorphic fxn.

(want convergence in Novikov ring ...)

What is the action?

$A \in GL(n, \mathbb{Z})$; $y \longmapsto Ay$

$\sum a_k z^k$ converges in U $\iff$ $\sum a_{A^T k} z^k$ converges in AU

k multi-index

$b \in \mathbb{R}^n$, $y \longmapsto y+b$

$$\sum a_k z^k \Longleftrightarrow \sum a_k T^{-\langle k, b\rangle} z^k$$

conv. in U

Need to see that semi-direct product acts on them?

<u>Exercise</u>.

Now we've defined $\mathcal{O}_Y$.

coh $\mathcal{O}_Y$-mod

(Paul: in the simplest case, take the circle, we've defined the analytic space associated to the Tate family of elliptic curves. - sheaves correspond to something like). over Spec Λ --

want $\mathcal{F}(X) \longrightarrow \mathcal{O}_Y$-mod

Consider Lagrangian submanifolds of X s.t.

$p|_L : L \to Y$ is an unramified covering.

think of a section or multisection.

Call resulting category $\mathcal{F}_{unram}(X)$.

$L \in \mathcal{F}_{unram}(X)$. $\boxed{\text{First, suppose } p|_L : L \to Y \text{ 1:1.}}$

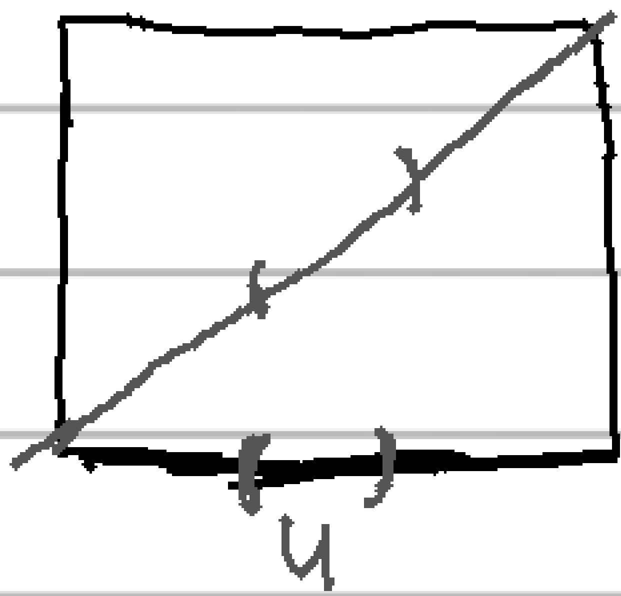
$F(L)$ ~ locally free $\mathcal{O}_Y$-module.

Over a sufficiently fine open covering $\{U_i\}$ of Y,

$$F(L)(U_i) \simeq \mathcal{O}_Y(U_i)$$

choice of isomorphism corresponds to choice of $f \in C^\infty(U_i)$ s.t. $L \cap p^{-1}(U_i) =$ graph of df mod $T^*_{\mathbb{Z}} Y$.

Ex:



Y

Change $f \longmapsto f + \ell$, where $d\ell$ is a section of $T^*_{\mathbb{Z}} Y$ $\Rightarrow$ multiply by $\exp(\ell) := T^c \prod z_i^{m_i}$

i.e. $\ell = c + \langle m, y \rangle$ $\qquad$ $\Uparrow$ $\mathcal{O}_Y^*$.

$\nwarrow$ integral

If $p: L \longrightarrow Y$ is <u>not</u> 1:1, take direct sum locally of sheaves associated to each sheet.

<u>ex</u>:

Ex: $Y = \mathbb{R}^n / \mathbb{Z}^n$, $X = \mathbb{C}^n / \mathbb{Z}^{2n}$, $c_1 = 0$.
abelian variety.

L_1 = section given by df, f = non-deg. quadratic form on $\mathbb{R}^n$ s.t. df takes integral values on $\mathbb{Z}^n$.

L_0 = 0-section $CF^*(L_0, L_1)$ concentrated in deg = index of quadratic form f.

Would like to see that

$$\mathrm{rk}\, HF^*(L_0, L_1) = \mathrm{rk}\, \mathrm{Ext}^*(F(L_0), F(L_1))$$

Concretely:

$F(L_0) = \mathcal{O}_Y$, b/c L_0 = graph df for $f \in C^\infty(Y)$ (e.g. $f=0$)

$HF^0(L_0, L_1) = \Lambda$

$HF^1(L_0, L_1) = 0$

$$\mathrm{Ext}^0(\underset{\mathcal{O}_Y}{F(L_0)}, F(L_1)) = \Lambda$$

$$= H^0(F(L_1)).$$

Suppose $\sum a_k z^k$ is some section.

In passing from neighborhood of 0 to a nbhd of 1,

shift: $y \mapsto y+1$.

difference

$$\tfrac{1}{2} y^2 \to \tfrac{1}{2}(y-1)^2 = \tfrac{1}{2} y^2 \boxed{-y+\tfrac{1}{2}}$$

$$\sum a_k z^k \to \sum a_k T^{-k} z^k \to \sum a_k T^{-k} z^{k+1} T^{1/2}$$

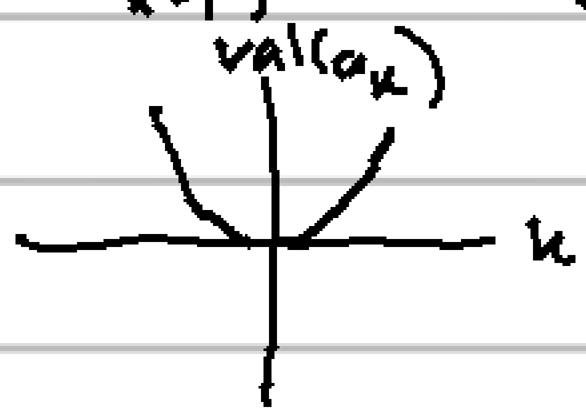
multiplying by exp(difference)

In order to define global section, need

$a_{k+1} = a_k T^{-k} T^{1/2}$ $\rightsquigarrow$ gives recurrence determining all coeffs from a single one, say a_0.

So $h^0(F(L_1)) \le 1$.

$\mathrm{val}(a_{k+1}) = \mathrm{val}(a_k) - k + 1/2$



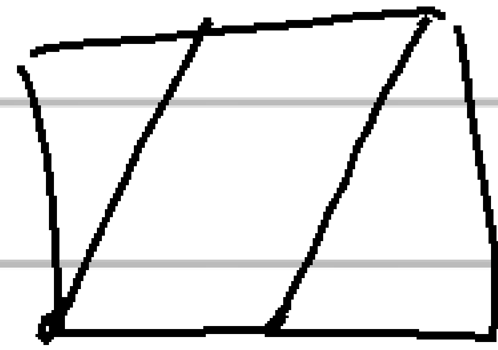
$\mathrm{val}(a_k) \sim k^2$

$\Rightarrow$ convergence

get a global section

$h^0(F(L_1)) = 1$

In this example



, get

recurrence $a_k \sim an+z$, so

2 degs. of freedom.

Note: $\exp(-y+\frac{1}{2}) = Z^{-1}T^{1/2}$.

$\mathcal{F}_{unram}(X) \to \mathcal{O}_Y\text{-mod}$ embedding,

When X torus, $\mathcal{F}_{unram}$ will split-generate.

(full subcat. of)

$D^\pi \mathcal{F}_{unram}(X) \simeq$ dg enhancement complexes of $\mathcal{O}_Y$-modules.

If X algebraic, supposed to get everything.

There's some kind of mirror torus fibration picture underlying it.