Talbot, Day 5 Talk 3: Richard Eager,

Matrix Factorizations & HMS

A-Model		B-Model
depends on Kähler moduli	$\xleftrightarrow{\text{HMS}}$	depends on cx structure moduli
A-branes		B-branes
Fuk		$D^b(\mathrm{Coh}\,X)$ or $MF(W)$

Nick's talk on Vanishing Cycles

$DF^{\to}(\check{X}, W) \longleftrightarrow D^b(\mathrm{Coh\ Fano}\ X)$

$\mathrm{Fuk}(\mathrm{Fano}\ X) \longleftrightarrow MF(X, W)$

Motivate category of matrix factorizations:

cf. Sarah's/ Parker's talk.

X affine, e.g. $\mathbb{C}^n$

$\underline{C}(X)$

objects: bounded $\mathbb{Z}$-graded complexes of projective objects in $\mathrm{Coh}(X)$

Morphisms: Regard as morphisms of $\mathcal{O}_X$-mod

DGn

$$H^0(\mathrm{Tw}\,A) \xrightarrow{\sim} D^b(\mathrm{Coh}\,X)$$
$$\overset{\shortparallel}{C(X)}.$$

Now we want to introduce a category

$C(X, W, w_0)$ LG model

$\shortparallel$ X smooth affine space

$DG_{w_0}(W)$ W holomorphic $X \to \mathbb{C}$

$w_0 \in \mathbb{C}$, WLOG $w_0 = 0$

(crit value of W)

Objects:

Pairs of f.g. projective $\mathcal{O}_X$-modules

$\bar{E} = (E_1, E_0)$

Morphisms $\mathbb{Z}/2\mathbb{Z}$-graded dg Category

(Paul: Shouldn't objects have differentials? Yes.)

$\mathrm{Hom}(\bar{P}, \bar{Q}) = \bigoplus_{i,j} \mathrm{Hom}(P_i, Q_j)$

Pair$_{w_0}(W)$

$DB_{w_0}(W)$ ~ Triangulated category

Differential: $f \in \mathrm{Hom}(P, Q)$

$Df = q \circ f - (-1)^{\overset{\text{deg.}}{k}} f \circ p$

Pair$_{w_0}(W)$

object $\bar{P}$ $\quad P_1 \underset{p_0}{\overset{p_1}{\rightleftarrows}} P_0$

$p_0 \circ p_1 = p_1 \circ p_0 = W \cdot \mathrm{Id}$.

(has to do w/ ∂^2 = central elt.)

(Pair_{w_0} is what we'd like to kill, DB is our category).
morphisms

Idea: Objects are curved but right morphisms have $D^2 = 0$, like Max's talk

$$\begin{tikzcd}
\bar{P} \arrow[d, "\bar{f}"] & P_1 \arrow[r, shift left, "p_1"] \arrow[d, "f_1"'] \arrow[dr, "t"] & P_0 \arrow[l, shift left, "p_0"] \arrow[d, "f_0"] \\
\bar{Q} & Q_1 \arrow[r, shift left] \arrow[ur, "s"] & Q_0 \arrow[l, shift left]
\end{tikzcd}$$

Morphisms in Pair

$\mathrm{Hom}(\bar{P}, \bar{Q})$

homogeneous deg 0 morphisms which commute with P.

Analog of null-htpic chain maps

$f_1 = q_0 t + s p_1$

$f_0 = t p_0 + q_1 s$,

$\left\{ \begin{array}{l} \underline{\text{Morphisms in } DB_{w_0}(W)} \\ \text{Morphisms in Pair modulo null htopic} \end{array} \right.$

Triangulated structure:

- Translation functor [1]
- Distinguished triangles satisfy axioms

(we know this comes from a dG cat. which is just triangulated).

Translation functor:

$$\tau : \bar{P} \longrightarrow \bar{P}[1]$$

$$\bar{P} = \left(P_1 \underset{p_0}{\overset{p_1}{\rightleftarrows}} P_0 \right)$$

$$\bar{P}[1] = \left(P_0 \underset{-p_1}{\overset{-p_0}{\rightleftarrows}} P_1 \right)$$

Note: [2] = Id functor.

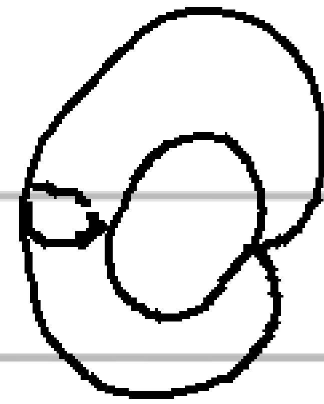
(In general, if $w_0 \neq 0$, have $\cdots = (W - w_0)\mathrm{Id}$)

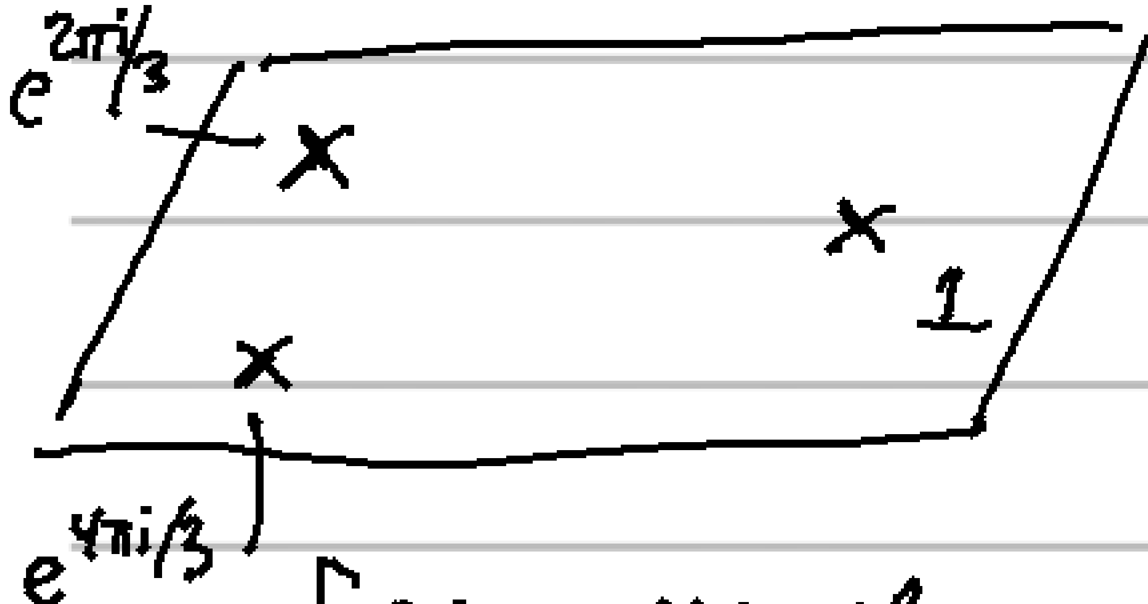
Examples: (HMS for $\mathbb{CP}^2$).

$\mathbb{CP}^2 \overset{\text{HMS}}{\longleftrightarrow}$ LG model $(\mathbb{C}^*)^2 \ni (x, y)$

$$W = x + y + \frac{1}{xy}$$

Recall picture:





Focus on $x=y=1$, $w_0=3$.

Shift to origin: $\begin{cases} x = u+1 \\ y = v+1 \end{cases}$ $\quad W = w_0$

$\Rightarrow xy(x+y-3)+1=0$

$(uv)(u+v-1) + (u+v)^2 = 0$

$\rightsquigarrow$ Quadratic in u, v + higher

u^2+v^2 + higher order

Holomorphic Morse lemma:

$\rightsquigarrow$ can kill cubic terms.

keep going push things off to ∞, & get:

LG

$W = u^2 + v^2$ on $\mathbb{C}^2$

$DB_{w_0=0}(W) =: K(n)$

$W = x_1^2 + \cdots + x_n^2$

Thm: There's an equivalence of categories between

$K(n)$ and $Cl_{mod}(n)$

complex Clifford algebra

Point: easier to compute $Cl_{mod}(n)$

(Correspondence is not hard)

- Mirror of the structure sheaf of a point (critical value of W, w_0)

$\rightsquigarrow$ SYZ $\rightsquigarrow$ torus in $\mathbb{C}P^2$.

Clifford torus (from Rui's talk).

3 local systems specified by their holonomies

$e^{2\pi i k/3}$

expect

$HF_\lambda(L, L)$ non-vanishing only for special holonomy,

$\lambda = e^{2\pi i/k}$.

Bonus:

Action of $Cl(2,\mathbb{C})$ on HF.

Stronger statement: $HF(L,L) \cong Cl(2,\mathbb{C})$

$\mathbb{Z}_2$ coeffs, $\mathbb{Z}_2$-graded v.s.

Comment:

This is a deformation

$$\Omega^*(L) \cong \Lambda^*(V^\vee)$$

$\nwarrow$ 2 dim

Grassman algebra

(so deformation took us to a Clifford algebra)

Graded Matrix Factorizations

If W quasi-homogenous

$$W(e^{i\lambda q_i} x_i) = e^{2i\lambda} W(x_i) \text{ for } \forall \lambda \in \mathbb{R}$$

$A = \bigoplus_i A_i$ graded $\longleftrightarrow$ vanishing of Maslov class.

Orlov's:

$$D^{gr}_{sg}(A) := D^b(\text{gr-}A) \big/ D^b(\text{gr proj-}A)$$

$\rightsquigarrow$ Floer cohom. can be $\mathbb{Z}$-graded.

f.g.

Orlov: $D^{gr}_{sg}(A) \longrightarrow D(\mathbb{Z}\text{gr-}A)$ $\rightsquigarrow$ using semi-orthogonal decomp.

Thm (Serre): $D^b(qgr A) = D^b(\mathrm{coh}\,\mathrm{Proj}(A))$

quasi-graded := allow q_i to be fractional.

Classical Koszul Duality

$$\Lambda = \Lambda(E) = \bigoplus_{i=0}^{n+1} \Lambda^i E$$

$\mathcal{M}(\Lambda)$ - category of $\mathbb{Z}$-graded modules over Λ

$\cup$

$\mathcal{F}$ — free graded Λ-modules

Thm: $\mathcal{M}^b(\Lambda)/\mathcal{F}$ is triangulated

$D^b(\mathrm{coh}\,\mathrm{Proj}\,A)$ $\quad A = \mathbb{C}[x_1, \dots, x_n]$

$D^b(\mathrm{Coh}\,\mathbb{P}^n)$ equiv.

deformed equivalence

$K(n) \cong Cl_{mod}(n)$ $\quad$ (Cliff. alg. = deform of grassman alg.)

$D^b(\mathrm{Coh}\,X) \cong \mathcal{M}^b(\Lambda)/\mathcal{F}$

when $W = 0$.

$K(n) \cong Cl_{mod}(n)$ is when

$$W = x_1^2 + \cdots + x_n^2$$

lose lot of information when passing to $K(n)$ from $D^b(Coh)$, has to do w/ localizing to boundary.

$x_i^2 \rightsquigarrow$ mass term

$x_i^3, x_i^4 \rightsquigarrow$ interaction term.

$W=0$ back to free theory, no localization