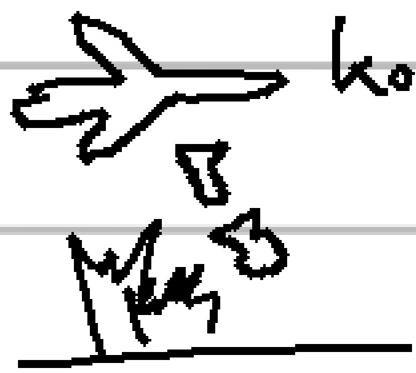
Day 5, Talk 4: Paul, Outlook.

2 weeks ago:



Categories of Constructible Sheaves

X topological space

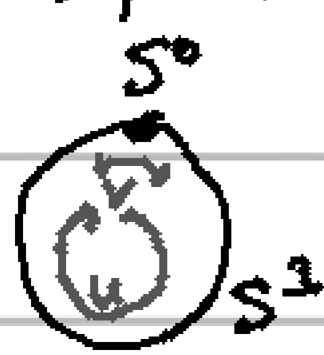
open strata

$\mathcal{S}$ stratification $\qquad \mathcal{S} = \{S\}$

$D^b(X, \mathcal{S})$ = derived category of complexes of sheaves of vector spaces whose cohomology is bounded & $\mathcal{S}$-constructible

In nice cases, these categories have quiver descriptions

Ex:

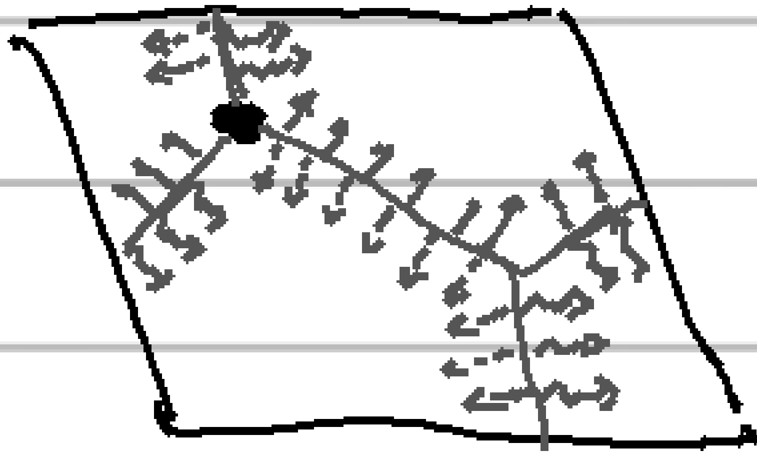


$F(V) \rightrightarrows F(U)$ corestriction maps.

$$\bullet \rightrightarrows \bullet \qquad D^b(X, \mathcal{S}) \cong D^b\mathrm{Coh}(\mathbb{P}^1).$$

Ex: (Bondal)

$X = T^2$



$\bullet = S^0$

$\cdot = S^1$

2-cell $= S^2$

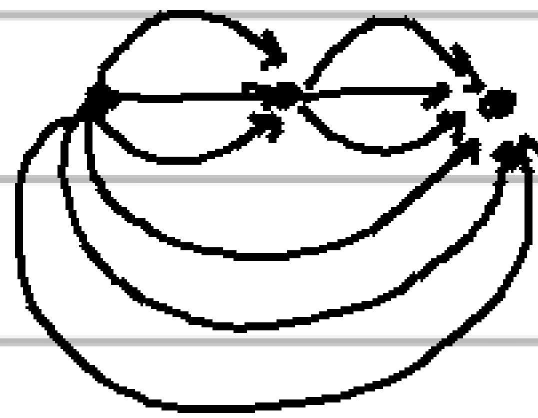
3 co-restrictions:

$\rightarrow$

$\dashrightarrow$

$\rightsquigarrow$

(b/c can't go through S^0)



up to sign, this is the Beilinson quiver for $\mathbb{P}^2$.

(supposed to use sheaves of orientations instead of constant sheaves?).

("co-amoeba picture" of vanishing cycle?)

Ex: $f: X \to \mathbb{R}$ Morse fcn. + generic metric.

Let $\mathcal{S}$ be the stratification by unstable mfolds $W^u(X)$.

The associated directed dg category is the so-called Morse category $\mathcal{C}$

$\mathrm{Ob}\,\mathcal{C}$ = critical pts. $x \in \mathrm{Crit}\, f$

$\hom_{\mathcal{C}}(x,y) = C_{-*}(\overline{\mathcal{M}}(x,y))$

Ex: A triangulation of X gives rise to a stratification, and the associated category of constructible sheaves categorifies the cellular chain complex

(Nadler - Zaslow)

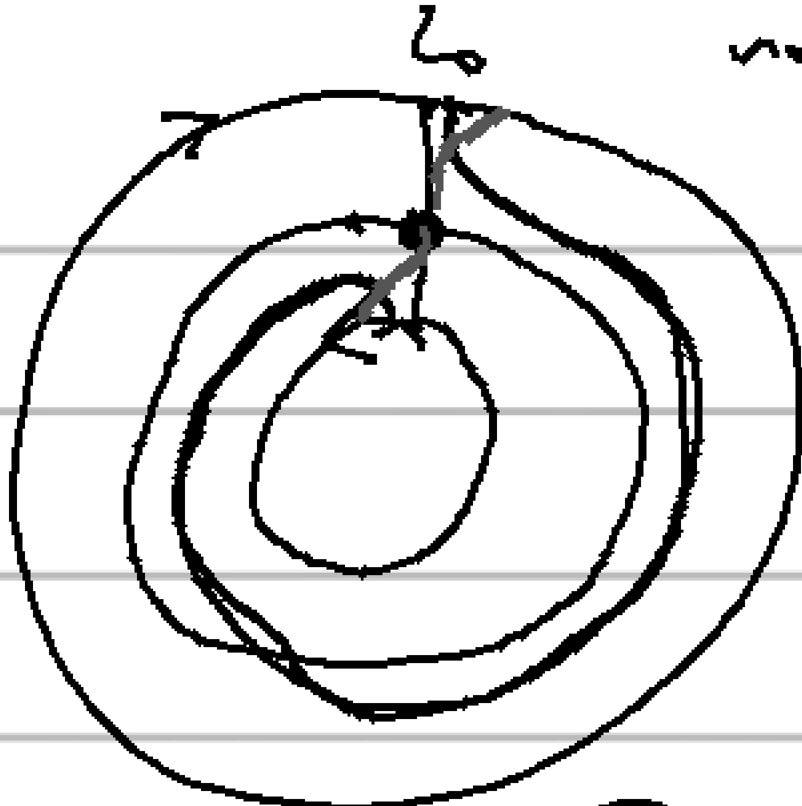
Thm: [Nadler] Take a smooth triangulation of a closed manifold X. Then there is a full exact embedding

$$D^b \mathrm{Fuk}(T^*X) \longleftrightarrow D^b(X, \mathcal{S})$$

(how useful is this? if X s.c., then left going only has things like zero sections. if not s.c., we're still figuring out implications.)

Idea: to each stratum S we associate a non-cpct Lagrangian submanifold $L_S \subset T^*X$ (smooth out Kashiwara's singular support construction.

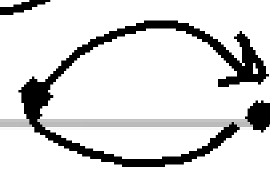
E.g.
$X = S^1$



non-compact conventions

$HF^*(L_0, L_1)$: perturb L_0 in + direction

$HF^*(L_0, L_1) = \mathbb{C}^2$



$= \bigoplus_{i,j} HF^*(L_i, L_j)$

$L \longmapsto \bigoplus_i HF^*(L_i, L)$ module over

(why non-compact perturbations like this? like Morse theory rel. boundary $\rightsquigarrow$ want function pointing in one direction or another, one gives you cohom. rel ∂, one gives you cohom.)

<u>Generalization</u>: Take M a symplectic manifold, exact, $\omega = d\theta$.

Take the vector field $Z \in C^\infty(TM)$ s.t. $\iota_Z \omega = \theta$

Liouville v.f. Assume that the flow of Z exists $\forall$ time & for time $\to -\infty$, it compresses M into some cpt subset.

E.g. T^*M, Z compressing flow.

Defn: The Lagrange skeleton $sk(M) \subset M$ is:

$$sk(M) = \{x \in M \mid \text{the forward } Z \text{ orbit of } x \text{ is relatively compact}\}$$

$sk(M) \hookrightarrow M$ is a homotopy equivalence and $sk(M)$ is a (usually singular) isotropic cycle.

Ex: $M = T^*X$, $X = sk(M)$ for standard θ.

Idea: Write $Fuk(M)$ as a category of sheaves on $sk(M)$.

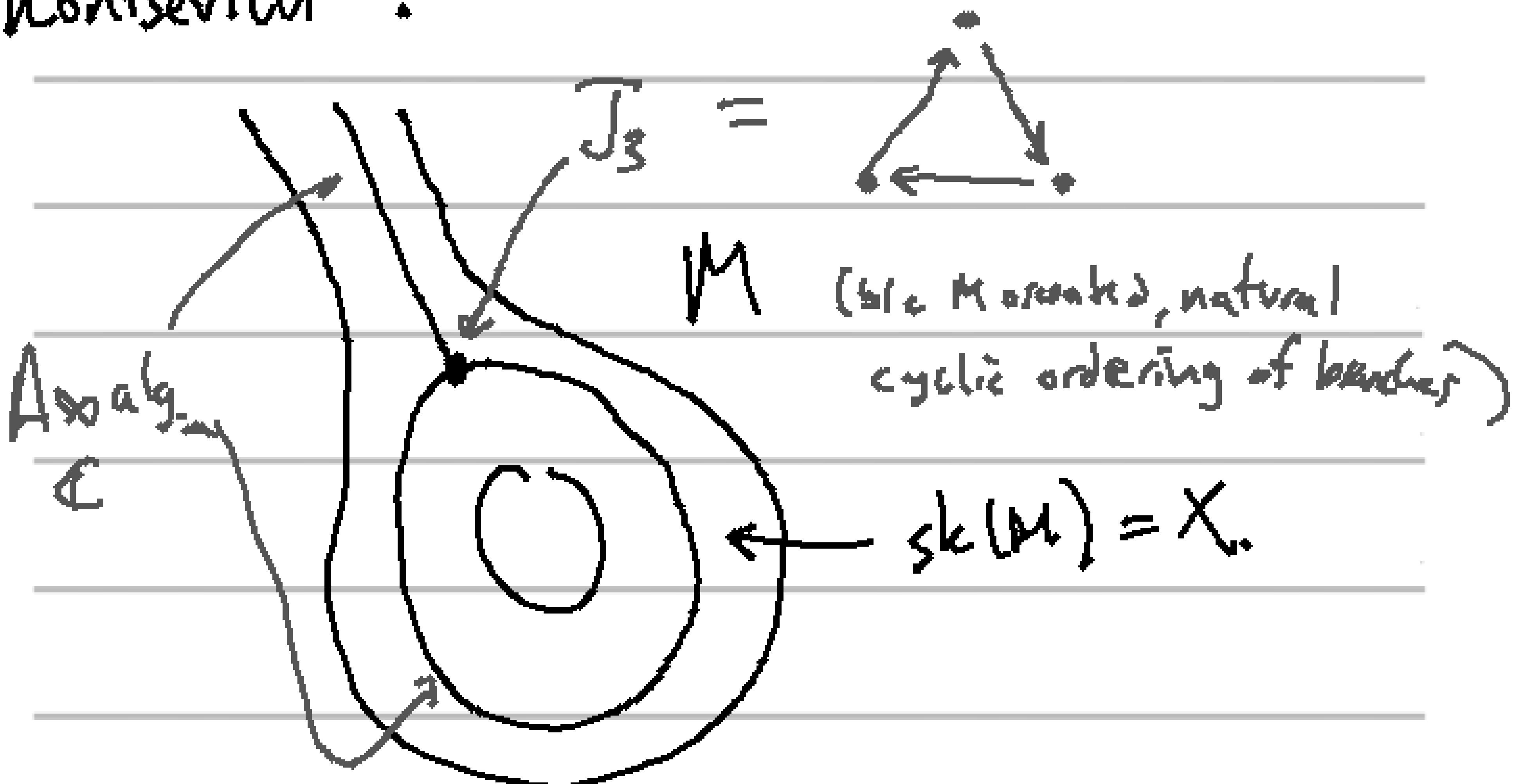
why crazy? $Fuk(M)$ inherently non-local!

Not crazy: has to do with exactness.

hol. strips have to be well controlled.

two pieces of evidence: Nadler's case, James's talk.

Kontsevich:



Equip X w/ a sheaf of A_∞ categories which is constructible w.r.t. the stratification.

Max: Is there a natural stratification on M?

(3 natural maps $\mathcal{T}_3 \to \mathbb{C}$, sending different pts to non-trivial obj.)

(or maybe just A_∞ algs where restriction maps are given by bi-modules?)

works "depressingly" well

Ex:

τ_3 here, i.e. get a triangle of chain complexes

Look at sheaves of A_∞ modules kronecker quiver.

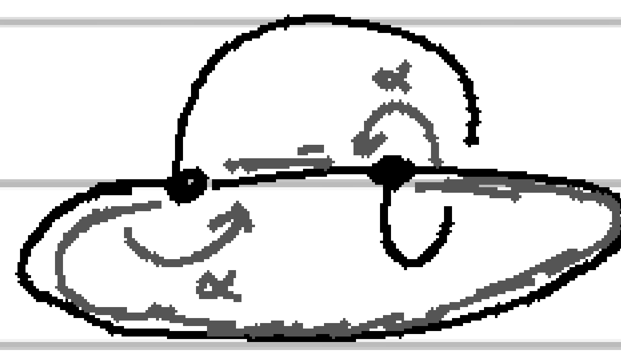
$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \end{tikzcd}$$

pts. correspond to gray lines on left

Kontsevich's picture of the Fukaya category of $\pi : \mathbb{C}^* \longrightarrow \mathbb{C}$, $\pi(x) = x + x^{-1}$, which is the mirror of $\mathbb{P}^1$.

Ex: $M = T^2 \setminus \text{Disc}$

$X = sk(M) =$



(quadruple isection points avoided here)

So modules over this sheaf of A_∞ algebras

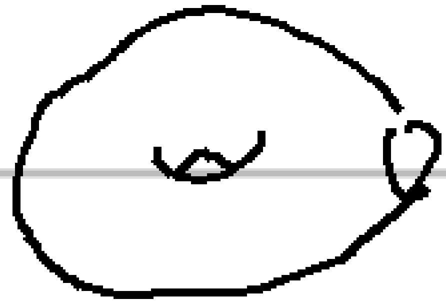
$$\begin{tikzcd} C \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & D \end{tikzcd}$$

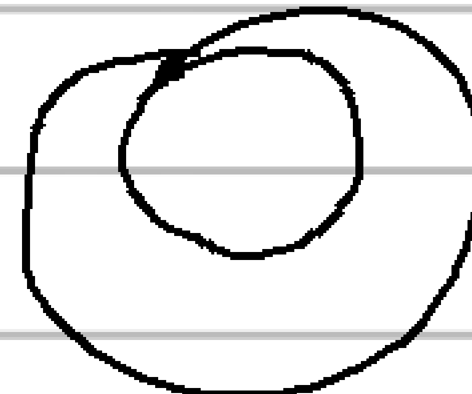
chain maps

C, D chain complexes

α, β chain maps.

together w/ a quasi-iso $\text{Cone}(\alpha) \cong \text{Cone}(\beta)$ (by picture above).

$M =$  is mirror to

$Z =$ 

(nodal elliptic curve)

(degenerate in sense of cpx structure, $\mathbb{P}^1$ w/ self isections)

$$D^{\pi}\mathcal{F}(M) \cong D^{\pi}\mathrm{Perf}(Z)$$

coherent sheaves which have bounded locally free resolution (they need to be cpct objects, i.e. homs out need to commute w/ colimits).

What's a sheaf on Z?

$$\mathbb{P}^1 \xrightarrow[\text{glues together } 0 \,\&\, \infty]{} Z$$

$$\boxed{\begin{array}{c}\mathrm{Perf}(Z)\\ \downarrow \\ \mathrm{Coh}(\mathbb{P}^1)\end{array}}$$

$$\begin{tikzcd} \mathrm{Perf}(Z) \arrow[r] \arrow[rr, bend left, "(*)"] & \mathrm{Coh}(\mathbb{P}^1) \arrow[r] & \mathrm{Mod}\left(\bullet \rightrightarrows \bullet \right) \end{tikzcd}$$

$$\mathcal{E} \longmapsto H^0(\mathcal{E}) \overset{1,\,z}{\rightrightarrows} H^0(\mathcal{E} \otimes \mathcal{O}(1))$$

two sections of $\mathcal{O}(1)$

(on chain complex level).

what's cone of 1? fiber of $\mathcal{E}$ at ∞.

cone of z? fiber of $\mathcal{E}$ at 0

images of $(*)$ come with an $\cong$ $\mathrm{Cone}(1) \simeq \mathrm{Cone}(z)$,

i.e. same as our picture of

In reality, more than T_3, T_4 happen.

But get a very large class of symplectic mfold.

There are moves between skeletons, & should be able to check: (A_5)

Check: genus 2 curve is non-formal, check it here.

Closed mfold: Locality destroyed by instanton corrections.

In general, things will develop on a (non-local) Nawikov parameter (close) manifolds.