7/17/2017, Abouzaid.

$X \to BU \qquad L \to BO$

$$\begin{array}{ccccccccc}
\Omega^2 X & \longrightarrow & \Omega^2(X,L) & \longrightarrow & \Omega L & & \Omega X & & \Omega(X,L) \\
\downarrow & & \downarrow{\scriptstyle A_\infty \text{ map}} & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}\times BU & \longrightarrow & \mathbb{Z}\times BO & \longrightarrow & O & & U & & U/O
\end{array}$$

lin. of $\bar\partial$ on space of discs

Conj:
$\Omega^2(X,L)$ lifts to an A_∞ alg. obj. in E_2-modules over $\Omega^2(X)$.

① hol. spheres (w/ framings) determine a curved framed E_2 deformation of $\Omega^2(X)^\nu_+$.

② discs define a deformation of $\Omega^2(X,L)^\nu_+$ as an A_∞ alg. over $\Omega^2(X)^\nu_+$.

③ same as a spherical cap.

$\Omega^2(X,L) \to \mathbb{Z}\times BO$ induces bdl. & Thom space.

Thm: $\Omega^2(X,L)^\nu$ & $\omega: \pi_2(X,L) \to \mathbb{R}$.

$\pi_2^+(X,L) \leftarrow \omega([u]) \geq 0$. $\rightsquigarrow$ $\Omega^2(X,L)^\nu_+$ components

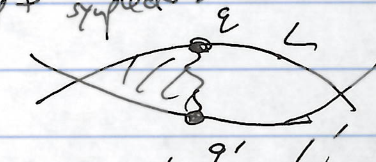
Analogy

com. algebra	sympl. top
field of fractions	$\pi_2(X,L)$
ring w/ valuation	$\pi_2^+(X,L)$ (group ring thereof over $\mathbb{R}$:)
maximal ideal	$\pi_2^{>0}(X,L)$
residue field:	$\pi_2^{=0}(X,L)$

If $\omega|_{\pi_2(X)} \equiv 0$.

① moduli of discs give a curved A_∞ deformation of $\Omega^2(X,L)^\nu_+$

② cat. of operadic modules is an invt. of (X,L) up to symplecto:
(to get: $\Omega\,\mathrm{Symp}(X) \longrightarrow \mathrm{Aut}$).



③ L, L' Ham. isotopic: then $\Omega^2(X,L,L')^\nu$ (Novikov-ly) defines a bimodule inducing an equivalence of module categories after localization at maximal ideal (& map from $\Omega\mathrm{Ham}(X) \longrightarrow \mathrm{Nat}(\mathrm{Id},\mathrm{Id})$).

Rothenberg–Steenrod, Dwyer–Greenlees–Iyengar:

$$L^{-TL} \simeq \mathrm{End}_{\Sigma^\infty_+ \Omega L}(\mathbb{S}).$$

$$\| \qquad DL = \mathrm{Fun}(L, \mathbb{S}).$$

$L \to pt$ makes $\mathbb{S}$ a module over ΩL.

key: make sure ~~L^{-TL} …~~ $\Omega^2(X,L)^\nu \simeq \Sigma^{-\dim \nu}\Omega^2(X,L)$.

to get back to L^{-TL} instead of a twisted version.

Condition: $\Omega^2(X,L) \to \mathbb{Z}\times BO \to BGL_1(\mathbb{S})$

$\Omega^2(X,L)^{\nu}$ $\uparrow$ cancel, so instead get $\Omega^2(X,L) \to \Omega(L)$.

$BGL_1(\mathbb{S}) = BF$ = classifying space of spherical fibrations

$\Omega^2(X,L) \to \mathbb{Z}\times BO \to BGL_1(\mathbb{S})$ compatible w/ products (E_1-map)

deloop: $\Omega(X,L) \to B^2 GL_1(\mathbb{S})$ (**)

then can compatibly trivialize tangent bundles of moduli of discs, as stable spherical fibrations, rel. the pull back of TL under evaluation

$$\mathcal{M}_1(L) \xrightarrow{ev} L$$

want: $ev^*(TL) \underset{\boxed{\text{spherically equivalent}}}{\simeq} T\mathcal{M}_1(L)$. points want spherical equivalence

(aspherical case).

Conj: L^{-TL} admits an (ungraded) curved A_∞ deformation whenever (**) is nullhomotopic.

Note: homotopy groups of $GL_1(\mathbb{S})$ are finite in each degree. L, X are finite complexes,

$\Rightarrow \Omega(X,L) \to B^2 GL_1(\mathbb{S})$ nullhomotopic after inverting finitely many primes.

$\Rightarrow$ get stable homotopy type except for finitely many p! (p = primes appearing in $\pi_k^{st}(\mathbb{S})$ up to $k = 2n+3$)

~~sphere bundles~~ non-aspherical case: moduli spaces are orbifolds.

Ravenel: fix prime p, $n \in \mathbb{N}_{>0}$, $K(n)$ height-n Morava theory.
Then, $K(n)$-oriented orbifolds satisfy Poincaré-duality w/ respect to $K(n)$.

e.g.,
$$K(n)^*(pt/G) \leftarrow K(n)_*(*/G)$$
$\uparrow$ [Ravenel]: finite $K(n)$-module

G finite group

In this case, study map
$$\Omega^2(X,L) \to BGL_1(K(n))$$ null-htpic? (if $p \geq 3$ and L is Pin),
L has (a framed) Floer-htpy type.

which vector bdles have a Thom iso. theorem? (all if $p \geq 3$)

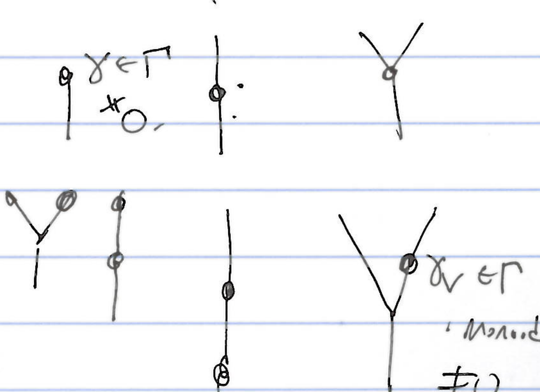
(L Pin $\Rightarrow$ $TM_1(L)$ orientable).

uses Thm: $\checkmark$ spherical fibration w/ ... for an oriented v.b. E, $\Rightarrow$ orientation (in whatever cohomology theory).

$L = \Delta$: true for all p. ("cplx vector bdles always have a Thom isomorphism").

O(0) O(1) O(2)

0 1 2



$K(n)$ theories are not E_∞ spectra
$$\left[\begin{array}{ll} p=2 & A_\infty \;\; ?? \\ p>2 & E_2 \end{array}\right].$$

Lemma: the ... A_∞ operad as algebra

"boundary conditions" - a type of module over this operad algebra?

$\Omega^2(X,L)^\vee_+$ γ_v' γ_v operad. A_∞ algebra. $\Omega^2(X,L)^+$ moduli ... (aspherical)

$\downarrow$ forget

... A_∞ ... graded by pos. part of ... $\pi_2(M,L)$

Then, essential modules are $\mathcal{O}_p$ algebra $\Omega^2(X,L)^\vee_+$.

Assume can undo this issue if X aspherical!

have $\Omega^2(X) \to \Omega^2(X,L) \to \Omega^1(L)$

"bounding cochains" ($\mathbb{Z}$-graded)

$\leftrightarrow$ local systems on L satisfying $\boxed{\text{M-C eq'n}}$ $\leftrightarrow$ modules over ΩL

which define to ~~[illegible]~~ curved modules

$\Omega^2(X,L)^\vee_0$ spectrum given by level of ev. map

$\Omega^2(X,L)^\vee_+ \to \Omega^2(X,L)^\vee_0 \cdot \mathbb{Q}$ mod $k[[t]]$ $\leftarrow$ Mod. supported at 0

mod $k((t,t^{-1}))$ $\quad \|$ ($k[[t]] \to k$, $t \to 0$)

Localization sequence:

mod k $\xhookrightarrow{i_*}$ mod $k[[t]] \twoheadrightarrow$ mod $k((t))$

$i : k[[t]] \xrightarrow{t \mapsto 0} k$.

Defn Floer htpy cat. of (X,L) = $\Omega^2(X,L)^\vee_*$ mod $/$ $\Omega^2(X,L)^\vee_0$-mod.

If X is a cotangent bundle T^*Q, Q = zero section, then

$$\Omega^2_*(T^*Q, Q) \simeq *.$$

fixed basepoint

got copy of ΩQ for every intersection point.

+ discs

$\Rightarrow \mathcal{P}_* Q = *$.

$\mathcal{W}(T^*Q) = C_*\Omega Q$-mod.

like otherwise X, $\Omega_2(X,L)$ is like "fibre of a point".

Moving basepoint: recover usual $\mathcal{W}(T^*Q)$.