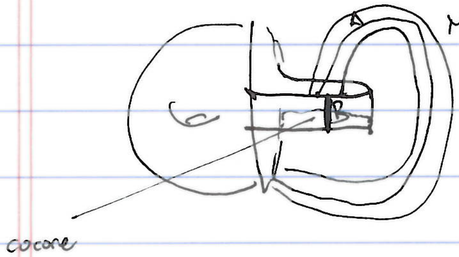
S. Kupers, Topological manifolds II

1) smooth transversality

2) normal microbundles

3) microbundle transversality.

Last time: wanted a theory based on handles & we showed handle decompositions.



Motivation:

- Q: can one slide the second handle to avoid co-core & be in original bndry.

If can ensure ∂ core $A \cap \partial$ cocore $B = \emptyset$, can do this. This is an instance of transversality theory

- topological Pontryagin-Thom: want to take $f^{-1}(0)$ & have it be nicely cut out,

locally modeled on $\mathbb{R}^k \hookrightarrow \mathbb{R}^n$ as coord. plane.

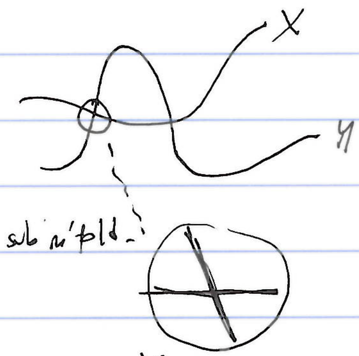
1) Smooth transversality. Def: M, N smooth m'flds, $X \subset N$ submanifold, $f: M \to N$ map.

Then, f is transverse to X ("$f \pitchfork X$") if, $\forall x \in X$, $\forall m \in f^{-1}(x)$

$$Tf(TM_m) + TX_x = TN_x.$$

Rmk: If f inclusion of submanifold, then M and X are transverse if their intersections locally look like generic affine m & x-dim'l planes in $\mathbb{R}^n$

- if $\dim M + \dim X < \dim N$, then $f \pitchfork X$ iff $f(M) \cap X = \emptyset$.



- implicit fn. theorem $\Rightarrow$ if $f \pitchfork X$, then $f^{-1}(X) \subset M$ is smooth sub m'fld.

Lemma: Every $f: M \to N$ can be approximated by a htpic smooth $\tilde{f}$ transverse to X.

2) Normal microbundles: Smooth $\pitchfork$ can be rewritten in terms of normal bundles.

$f: M \to N$ is $\pitchfork X$ iff $\forall x \in X$ $\forall m \in f^{-1}(x)$, we have

$$TM_m \xrightarrow{Tf} TN_x \longrightarrow TN_x / TX_x =: (\nu_X)_x \quad \text{is surjective.}$$

Recall $\nu_X = TN|_X / TX$ vector bundle over X.

Generalization to topological manifolds necessary; but it will be a microbundle not vector bundle.

Def: An <u>n-dim'l microbundle</u> over B is a triple $\xi = (E, i, p)$ of

- E : space "total space"
- $i : B \to E$ "zero section"
- $p : E \to B$ "projection", satisfying

(i) $p \circ i = \mathrm{id}_B$ ("i is a section of p")

(ii) $\forall\, b \in B$, $\exists$ open nhoods $U \subset B$ of b, $V \subset p^{-1}(U) \subset E$ of $i(b)$ and a homeomorphism $\phi : \mathbb{R}^n \times U \to V$ such that these diagrams commute:

$$
\begin{tikzcd}
\mathbb{R}^n \times U \arrow[r, "\phi"] & V \\
U \arrow[u, hook, "\{0\} \times -"] \arrow[r, equal] & U \arrow[u, hook, "i"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{R}^n \times U \arrow[r, "\phi"] \arrow[d, "\pi_2"'] & V \arrow[d, "p"] \\
U \arrow[r, equal] & U
\end{tikzcd}
$$

<u>Rmk:</u> E need not be htpy equivalent to B! (one ex: a nhood of $\underline{0}$ in a vector bundle, but w/ arbitrary topology added outside, so ~~a ... "..." name?~~ how p, i).

Say $\xi = (E, i, p)$ and $\xi' = (E', i', p')$ over B are <u>equivalent</u> if there are nbhds $W \subset E$ of $i(B)$ & $W' \subset E'$ of $i^*(B)$ & a homeomorphism $\varphi : W \xrightarrow{\cong} W'$ compatible w/ data.

<u>examples:</u> (1) every vector bundle is a microbundle.

(2) trivial microbundle: $(\mathbb{R}^n \times B, i, \pi_2)$

(3) If M is a topological manifold, then $(M \times M, \Delta, \pi_2)$ is an n-dim'l microbundle, the "tangent microbundle".

(to check (ii), a local condition, reduce to the case of $\mathbb{R}^n$).

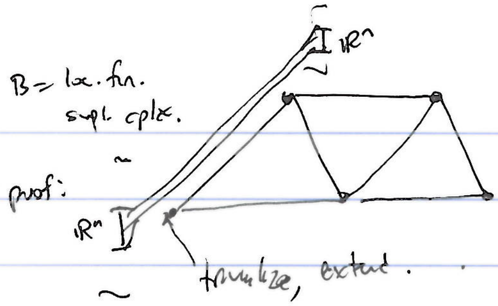
(if M is smooth, this is equiv, as microbundles, to TM)

~~They~~ Microbundles behave a lot like vector bundles. (E.g., every microbundle over a paracompact contractible B is $\simeq$ a trivial one. (so, htpy property, can try to classify them, etc.))

<u>Thm</u> (Kister–Mazur): Every n-dim'l microbundle over a sufficiently nice base (maybe "locally finite CW cplx" "ENR", etc.) is equivalent to a fibre bundle with fibers $\mathbb{R}^n$ and structure group $\mathrm{Top}(n) := \mathrm{Homeo}(\mathbb{R}^n, 0)$ (or in $\mathbb{R}^n$?), with total space contained in E, unique up to equivalence (or rather, fibrewise embeddings).

(a stronger version is that it's the "up to contractible choice" ~~...~~ equivalence of classifying spaces).

words about proof:



Rmk: can't always find a disc bdle in a microbdle! [Browder] (have to add an $\mathbb{R}$ to do so?)

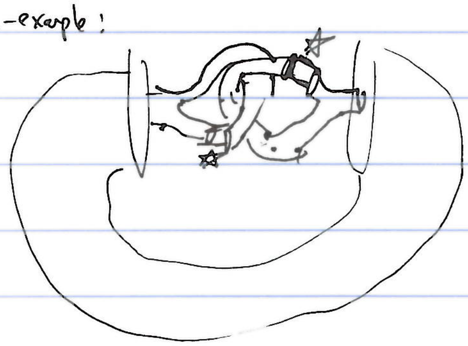
Def: A (locally flat) submanifold X of a top. m'fold N is a closed subset s.t.

$\forall\, x \in X$, $\exists$ an open $U \subset N$ containing x s.t. $(U, U \cap X) \underset{\text{homeo.}}{\cong} (\mathbb{R}^n, \mathbb{R}^x)$

examples: • if $X \subset N$ smooth manifold, then it is locally flat.

• non-example:

(a)

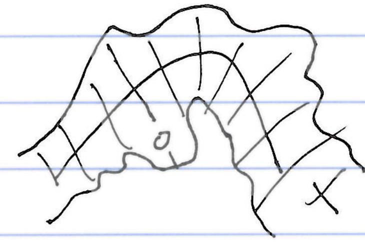


in ☆'s inductively reinsert this picture

[Alexander Horn sphere]

(b) [Lipshitz: cone on a knot in S^3].

Def'n: A normal microbundle ν to a locally flat sub'fld $X \subset N$ is an $(n-x)$-dim'l microbundle $\nu = (E, i, p)$ over X w/ an embedding of a nbhd of 0-section of E into N (extending $X \hookrightarrow N$).



(might not exist in general, or be unique if it exists!)

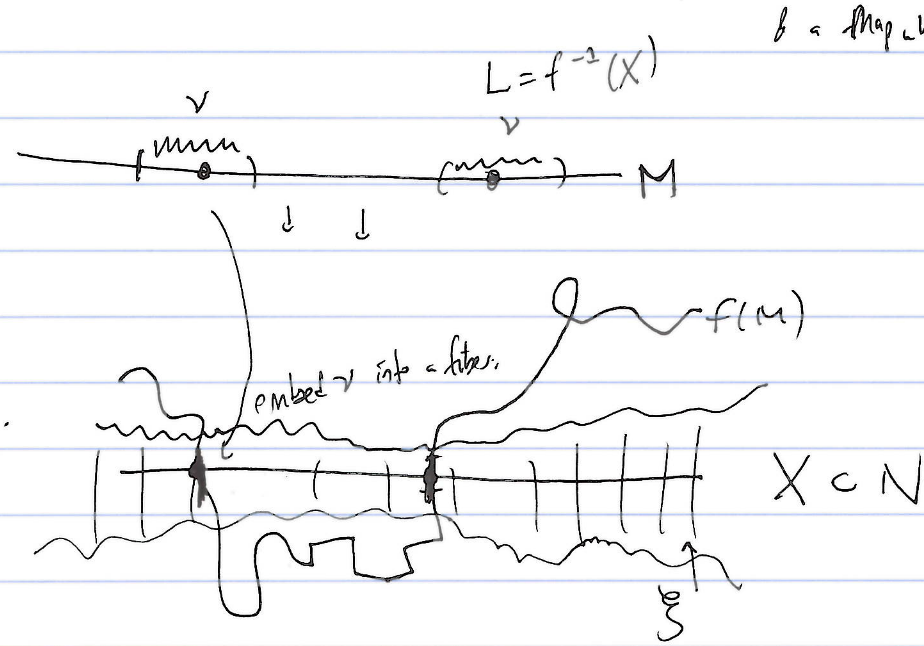
but stably, exist and are unique.

or when codim 1 or 2 or dim $X \leq 4$ (or N?)

3. Microbundle transversality:

Def: $X \subset N$ locally flat submanifold with normal microbundle ξ. Then, a continuous map $f: M \to N$ is microbundle transverse to ξ (at ν) if:

- $f^{-1}(X) \subset N$ is a locally flat submfld with normal microbdle ν.
- f induces an open embedding of a nbhd of 0-section of a fiber of ν into a fiber of ξ. (Or rather, for each pt in $f^{-1}(X)$, $\exists$ a chart & a map which is a fibrewise embedding).

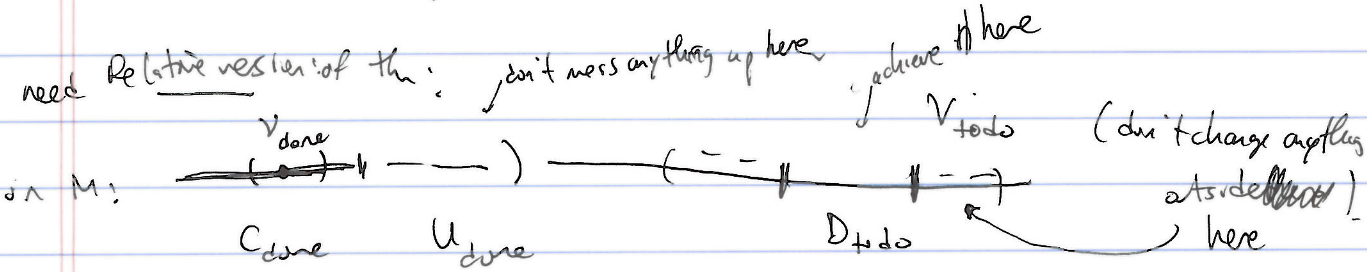


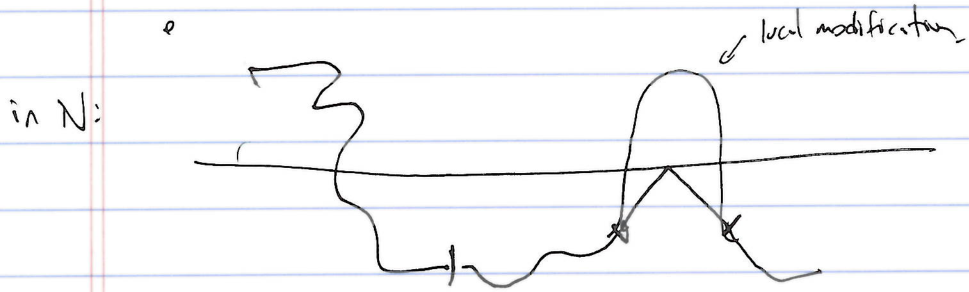
Thm: Let $X \subset N$ a locally flat submfld w/ normal microbundle $\xi = (E, i, p)$. If $m + x - n \geq 6$, any $f: M \to N$ can be approximated by a homotopic $\tilde{f}$ which is microbundle transverse to ξ. (actually, is true in all dimensions).

[Kirby-Siebenmann, Quinn]

↑ exceptional dimensions

need Relative version of thm:





so, try to do charts, where can put on smooth structures.

Step 1: $M \subset \mathbb{R}^n$ open

$X = \{0\}$

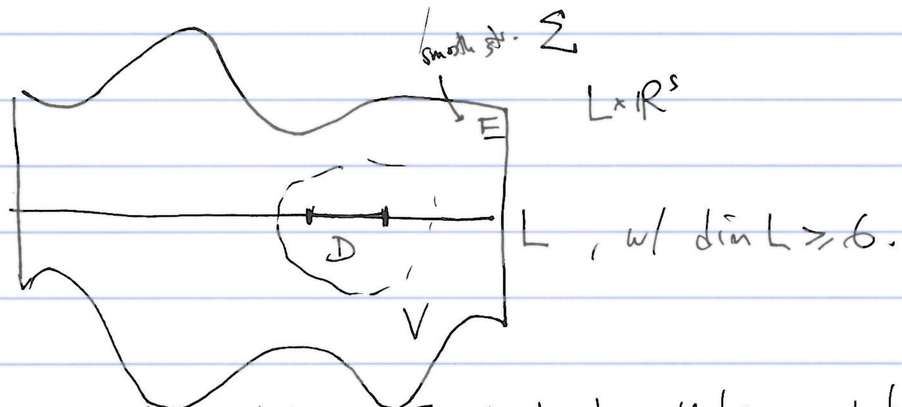
$N = \mathbb{R}^{r=n-x}$

$\xi = \mathbb{R}^r \times \{0\}$

We'll need:

using: Thm: (local product structure theorem):

Given:



$\exists$ a concordance of smooth structures starting at Σ and ending at one that is a product on V, rel $E \setminus V$.

---

apply Thm to: $L = f^{-1}(X) \cap U_{done}$, E = total space of an $\mathbb{R}^r$-bundle inside $E(\nu_{done})$

$s = r$, $D = L_{done} \cap C_{done}$, V some nbhd,

$\Sigma = E \subset M \subset \mathbb{R}^n$ to get smooth structure.

~~Requires~~ we'll need $E \subset L \times \mathbb{R}^r$ using (p, f)

at the end, L smooth & f smooth $\rightsquigarrow$ now use relative transversality for smooth maps.

Step 2: $M \subset \mathbb{R}^m$, X higher dim'l, but $\underbrace{\xi \text{ is trivializable}}_{\text{assume it's } X \times \mathbb{R}^r}$.

Now, $M \xrightarrow{f} X \times \mathbb{R}^r$ transverse to $X \iff M \xrightarrow{f} X \times \mathbb{R}^r \xrightarrow{\pi_2} \mathbb{R}^r$ transverse to 0.

(reduced to step 1)

Step 3: Induction over charts (ok b/c of the strongly relative version proven). ∎

(Rmk: in topological worlds, this ~~$\Rightarrow$~~ ⋔ for ~~maps~~ embedded m'folds (b/c embeddings not open, unlike smooth case)

Instead, bootstrap from PL world.