## S. Kupers III: Topological manifolds III

1) Thom spectra

2) Cobordism groups

3) Pontryagin–Thom construction

Goal: classification of topological manifolds of dimension $\geqslant 6$, up to cobordism, in terms of stable htpy theory.

### 1) Thom spectra.

For a topological group $G$, there is a htpy type $BG$ classifying principal $G$-bdles over nice (e.g., paracpct) spaces. Meaning,

$$\left\{\begin{array}{c}\text{principal } G\text{-bdles} \\ \text{over } B\end{array}\right\}\Big/\sim \;\xleftrightarrow{\;\cong\;}\; [B, BG]$$

$$[f^*\gamma] \;\longleftarrow\!\!\shortmid\; [f]$$

$BG$ carries a universal principal $G$-bdle $\gamma$, and

Ex: $G = Top(n) := Homeo(\mathbb{R}^n, 0)$. Then, $BTop(n)$ classifies principal $Top(n)$-bundles $\leftrightarrow$ $\mathbb{R}^n$-bdles w/ structure group $Top(n)$

$P \mapsto P \times_{Top(n)} \mathbb{R}^n$

(to go back, take fibrewise automorphisms)

Have $\cdots \to Top(n) \to Top(n+1) \to \cdots$ induces, using a functorial construction of $BG$:

$$\cdots \to \underset{\substack{\Theta_n \\ \text{univ. } \mathbb{R}^n \text{ bdle}}}{BTop(n)} \xrightarrow{i_n} \underset{\substack{\Theta_{n+1} \\ \text{univ. } \mathbb{R}^{n+1} \text{ bdle}}}{BTop(n+1)} \to \cdots$$

satisfying $i_n^* \Theta_{n+1} = \varepsilon \oplus \Theta_n$.

(trivially, product w/ $\mathbb{R}$ in each factor)

Def: A sequential tangential structure $\Theta$ is a functor $B : \underbrace{\mathbb{N}_{\leq}}_{\text{poset of natural numbers under } \leq} \longrightarrow Top$

together with a bundle $\Theta_n$ over $B_n$

& isos. $i_n^* \Theta_{n+1} \cong \varepsilon \oplus \Theta_n$.

(maybe only want/need these structures for $n \gg 0$).

e.g., arise from natural transformations $B \longrightarrow BTop := (BTop(n))_{n \geqslant 0}$ of functors $\mathbb{N}_{\leq} \to Top$

For any $\mathbb{R}^n$ bundle $\xi$ over $B$, we have the pointed space

$$Th(\xi) = \left\{\begin{array}{c}\text{fibrewise one-pt-} \\ \text{cptification of } \xi\end{array}\right\} \Big/ \infty\text{-section}.$$

(if base $B$ is cpct, just take 1-pt. cptification all at once)

**Example:** $\varepsilon^n$ trivial $\mathbb{R}^n$-bdle over $B$, then

$$Th(\varepsilon^n) = S^n \wedge B_+ ;\quad \text{more generally}$$

$$Th(\varepsilon^1 \oplus \xi^n) = \Sigma Th(\xi) \,..$$

Recall a (naive, or pre-) spectrum is a sequence of pointed spaces $E_n$ together with maps

$$\Sigma E_n \to E_{n+1}. \qquad \text{(if needed, spectrify later).}$$

**Def:** For $\Theta$ a sequential tangential structure, the Thom spectrum $M\Theta$ has $n^{th}$ space $(M\Theta)_n = Th(\Theta_n)$.

The maps $\Sigma Th(\Theta_n) \longrightarrow Th(\Theta_{n+1})$ are induced by $\varepsilon \oplus \Theta_n \cong i^* \Theta_{n+1}$ e.g.,

$$\Sigma Th(\Theta_n) = Th(i^* \Theta_{n+1}) \xrightarrow[\substack{\text{include} \\ \text{over} \\ B_n \to B_{n+1}}]{\text{fibrewise}} Th(\Theta_{n+1}).$$

**Example:** $B = (*)_{n \geq 0}$, (so $\Theta_n$ trivial always), then $Th(\Theta_n) = S^n$ and $S^1 \wedge S^n \longrightarrow S^{n+1}$ the std. isos, so $M\Theta = \mathbb{S}$.

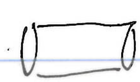
Spectra have stable htopy groups

$$\pi_n E = \operatorname{colim}_{k \to \infty} \pi_{n+k}(E_k).$$

2) Cobordism groups

**Def:** Two compact $n$-dim'l topological manifolds $M_0, M_1$ are cobordant if there exists an $(n+1)$-dim'l top. manifold $W$ (a cobordism) with $\partial W \underset{\text{homeo}}{\cong} M_0 \amalg M_1$. (Ex: this is an equivalence relation).

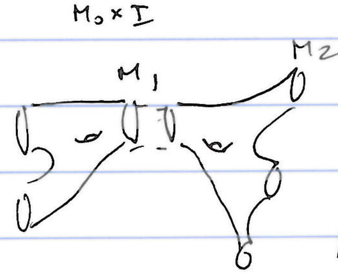
- symmetric ✓
- identity :



$M_0 \times I$ ✓

- associative :

glue:




So, define

$$\Omega_n^{Top} = \left\{ \begin{matrix} n\text{ dim'l} \\ \text{cpct. top. m'flds} \end{matrix} \right\} \Big/ \text{cobordism}$$

abelian monoid under $\amalg$.

$M_0 \times I$ is also a cobordism from $M_0 \amalg M_0$ to $\emptyset$,

$\Rightarrow -[M_0] = [M_0] \Rightarrow$ group.

**Example:** all 0-dim'l cpct. top. manifolds are finite disjoint unions of points & $(\cdot\ \cdot) \sim \emptyset$

$$\Rightarrow \Omega_0^{Top} \cong \mathbb{Z}/2$$

And similarly, all 1-mfld's are $\amalg$ O's & $(\bigcirc) \sim \emptyset$ by (disk).

$$\Rightarrow \Omega_1^{Top} \cong 0 \,..$$

and $\Omega_n^{Top} \cong \mathbb{Z}/2 \longleftrightarrow$ gen. by $\mathbb{RP}^2$ (b/c using our understanding of 2-mflds).

In fact, everything is 2-torsion, b/c there's a ring str. by $\times$, & unit for $\times$, $\circ$, is 2-torsion.

More general def'n of $\Omega_n^{Top, \Theta} \leftarrow \Theta$ structure on "stable normal bdle."

Lemma: Every cpct. top. manifold admits a locally flat embedding into $\mathbb{R}^s$ $s>>0$ (usual Whitney emb., using partition of 1)

Pf: $\phi_i : U_i \subset \mathbb{R}^n \to M$ charts, $i \in I$ finite

partition of unity $\eta_i : M \to [0,1]$, then define

$$\Psi : M \longrightarrow \mathbb{R}^{|I|(n+1)}$$
$$m \longmapsto (\eta_i(m), \eta_i(m)\,\phi_i^{-1}(m))_{i\in I}$$

(can always show locally flat). ☒.

Thm: (Brown) If $X \subset N$ locally flat, then $\exists$ $S >> 0$ depending only on $\dim X$, $\dim N$, s.t.

$X \times 0 \subset N \times \mathbb{R}^s$ for $s \geq S$ has a normal microbundle and, if $s \geq S+1$,

it's unique up to concordance (e.g., $\exists$ microbdle over $X \times I$, —).

Given $M^n$;
i) embed in $\mathbb{R}^S$
ii) find a normal microbundle in $\mathbb{R}^S$ (maybe increasing $s$)
iii) using Kister-Mazur, find an $\mathbb{R}^{s-n}$ $\underbrace{\text{bdle}}_{\nu_{s-n}}$ in normal microbdle.

Def'n: A $\Theta$ structure on $\nu_{s-n}$ is a $\mathbb{R}^{s-n}$-bundle map from $\nu_{s-n}/M$ to $\Theta_{s-n}/B_{s-n}$

(e.g., if $\Theta = \{*\}$) (needs to be an iso. when one pullsback!)

- A $\Theta$-str. on stable normal bdle of $M$ is an equivalence class of $\Theta$-str. on $\nu_{s-n}$ up to concordance & increasing $s$.

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If $W$ is an $(n+1)$-dim. cpct m'fold w/ bdry and $\Theta$-str. then $\partial W$ inherits a unique $\Theta$-str. on stable normal bundle. So, can define $\Omega_n^{Top,\Theta}$

3. Pontryagin-Thom theorem: $\Omega_n^{Top,\Theta} \cong \pi_n(M\Theta)$.

by taking $\Theta = \{*\}$

Rmk: $\Omega_n^{Top, fr} \cong \pi_n(\mathbb{S}) \cong \Omega_n^{Diff, fr}$

$\Rightarrow$ any top. m'fld w/ framing of stable normal bdle. admits a smooth str., up to cobordism. (in fact already true w/o cobordism, by embedding $M \hookrightarrow \mathbb{R}^s$, using $\underset{M \times \mathbb{R}^{s-n}}{U \supset M}$ ($U \cong M \times \mathbb{R}^{s-n}$) & prod. structure theorem ~~[illegible]~~ for large $n$).

Proof when $n \geq 6$, no $\Theta$ structures:

$\mathcal{C} : \Omega_n^{Top} \longrightarrow \pi_n(MTop)$ "Pontryagin–Thom collapse"

& $\mathcal{L} : \pi_n MTop \longrightarrow \Omega_n^{Top}$ "transverse inverse image".

For $\mathcal{C}$: $M$

i) embed it in $\mathbb{R}^s$

ii) pick normal microbdle.

iii) pick an $\mathbb{R}^{s-n}$-bdle. in normal microbdle.

iv) pick a classifying map $M \xrightarrow{\gamma_{s-n}} BTop(s-n)$.

Take: $S^s = \mathbb{R}^s \cup \{\infty\} \longrightarrow \mathbb{R}^s / \mathbb{R}^s \setminus E(\gamma_{s-n}) \cong Th(\gamma_{s-n}) \to Th(\Theta_{s-n}) \; =: c_s$;

& there's a commutative diagram (if construct carefully)

$$\begin{array}{ccc} S^1 \wedge S^s & \xrightarrow{S^1 \wedge c_s} & S^1 \wedge M(\Theta_{s-n}) \\ \downarrow & & \downarrow \\ S^{s+1} & \xrightarrow{c_{s+1}} & M(\Theta_{s+1-n}) \end{array}$$

, take colim to get

$\rightsquigarrow \pi_n(MTop) \ni \mathcal{C}(M)$. (& now check unchanged by cobordism).

For $\mathcal{L}$ $c \in \pi_n(MTop)$, (i) take a representative

$c_s : S^s \to Th(\gamma_{s-n})$.

(ii) make $c_s$ transverse to 0-section. (c.f., proof from last time, "globalized") (e.g., microbundle transverse), call the result $\tilde{c}_s$ (note 0-section has a canon normal microbdle.).

(iii) $\mathcal{L}(c) := \tilde{c}_s^{-1}(\text{0-section})$.

If change $c$ by htpy, make the htpy generic, get cobordism.

Upgrade to include $\Theta$-structures ...