J. Pardon I, Orbifolds (and their fundamental classes)

An orbifold is a "space" which is locally modeled on $\mathbb{R}^n/G$ ← finite group action for some finite group $G \circlearrowright \mathbb{R}^n$.

A (smooth) m'fold is a top. space M which is:

(1) Hausdorff

(2) locally homeomorphic to $\mathbb{R}^n$

(3) 2^{nd} countable + paracompact

together w/ a smooth structure: namely either:

(3) A collection of maps

$$\varphi : U \to M \qquad (U \subseteq \mathbb{R}^n \text{ open})$$

homeos onto image, covering M

w/ smooth transition maps, which is maximal — "mapping in property"

(3') A subsheaf "$C^\infty(M)$" $\subseteq C(M)$ which is locally isomorphic to $(\mathbb{R}^n, C^\infty(\mathbb{R}^n))$. — "mapping out property"

Orbifolds can be defined by a mapping in property, but not by a mapping out property (contrast w/ the def'n of the topological quotient $\mathbb{R}^n/G$).

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r] \arrow[d] & X \\
\mathbb{R}^n/G \arrow[ur, dashed] &
\end{tikzcd}$$

topologically

Main reason why defining orbifolds is subtle is that

$C^\infty(\mathbb{R}^n, \mathcal{O})$ is not a set but rather a groupoid for an orbifold $\mathcal{O}$.

(orbifolds form a 2-category instead of an ordinary category).

Recall: A groupoid is a category in which all morphisms are isomorphisms.

Ex: G group $\rightsquigarrow$ groupoid BG w/ single object $*$ and $\mathrm{Aut}(*) = G$.

Every groupoid is just equivalent to $\coprod_\alpha BG_\alpha$.

Rmk: An equivalence of categories is $F : \mathcal{C} \to \mathcal{D}$ s.t.

(1) $\mathcal{C}(a,b) \xrightarrow{\sim} \mathcal{D}(F(a), F(b))$ — injective on iso. classes

(2) $\forall\, d \in \mathcal{D}$, $\exists\, c \in \mathcal{C}$ s.t. $F(c) \cong d$. — surj. on iso. classes

Ex: $\mathcal{C} = \bullet\circlearrowleft$

$\mathcal{D} = \bullet\circlearrowleft \rightleftarrows \bullet\circlearrowleft$ are equivalent.

The following two statements force $C^\infty(\mathbb{R}^n, \mathcal{O})$ to be a groupoid:

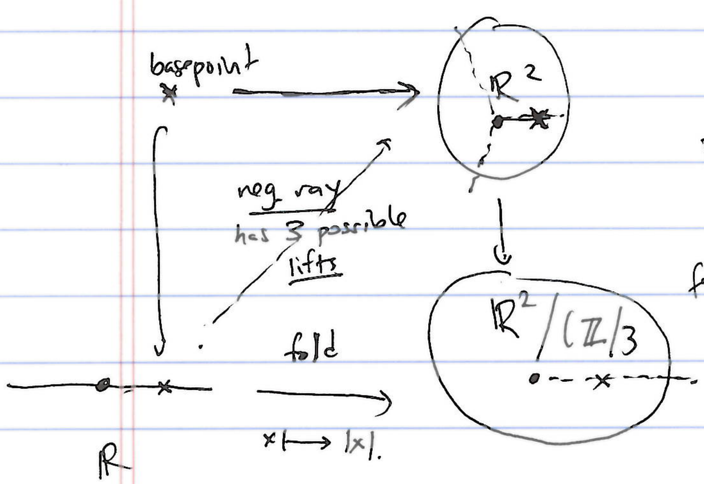
(1) Want $\mathbb{R}^n \longrightarrow \mathbb{R}^n/G$ to be étale (i.e., a covering space)

e.g.,
$$\begin{tikzcd} * \arrow[r] \arrow[d, hook] & \mathbb{R}^n \arrow[d] \\ M \arrow[r] \arrow[ur, dashed, "\exists!"] & \mathbb{R}^n/G \end{tikzcd}$$
(for simply connected M) ← smooth manifold.

(2) We want the functor $U \longmapsto C^\infty(U, \mathcal{O})$ to satisfy descent

(i.e., maps to $\mathcal{O}$ can be defined locally).

Why does this force groupoids on us? Consider the following example (in continuous category)



Problem! (think of morphisms as groupoids...!)

Solution: downstairs, there should be three possible fold maps: secretely at 0 $\exists$ a copy of $B\mathbb{Z}_3$,

& to specify a map, need a map from both halfs & a path in $B\mathbb{Z}_3$ between the two images.

in terms of descent: gluing data = glue restr. of maps to origin.

hence, there have to be three maps.

in terms of descent: choose map on two rays, & specify descent data for path

$\Rightarrow C^\infty(pt, \mathbb{R}^2/(\mathbb{Z}/3))$ contains $B\mathbb{Z}/3$.

Why are spaces $\mathcal{O}$ for which $C^\infty(\mathcal{U}, \mathcal{O})$ are groupoids useful?

Moduli space $\mathcal{M}_g$ satisfies a universal property:

$$\{\mathcal{U} \longrightarrow \mathcal{M}_g\} = \left\{ \begin{array}{c} E \\ \downarrow \\ \mathcal{U} \end{array} \text{ family of closed genus } g \text{ Riemann surfaces} \right\},$$

RHS (of iso. classes), regarded as a set, does not satisfy descent.

But, it does satisfy descent when regarded as a groupoid.

Def (Moerdijk): A proper étale groupoid is a ~~category~~ groupoid $\mathcal{C}$ where

$\mathrm{Ob}\,\mathcal{C}$ and $\mathrm{Mor}\,\mathcal{C}$ are smooth manifolds, and:

① $s, t : \mathrm{Mor}\,\mathcal{C} \longrightarrow \mathrm{Ob}\,\mathcal{C}$ are smooth and ~~proper~~ étale (meaning $T_xM \xrightarrow{\cong} T_{f(x)}N$ $\forall x$.) ("covering ~~space~~ map finite degree")

($\mathrm{Mor}\,\mathcal{C} \xrightarrow{(s,t)} \mathrm{Ob}\,\mathcal{C} \times \mathrm{Ob}\,\mathcal{C}$ is proper?)

② ~~oth~~ Composition is smooth.

Def: An orbifold is a proper étale groupoid. — proper = analogue of Hausdorff-ness

(think of this as giving a "presentation" of an orbifold, actually).

Warning: A map of orbifold $\mathcal{O} \longrightarrow \mathcal{O}'$ is not simply a "smooth functor of proper étale groupoids."!

(instead, define in terms of correspondences: $\begin{array}{c} Z \\ \cap \\ \mathcal{O} \times \mathcal{O}' \end{array}$ ~~④~~ closed sub proper étale groupoid which projects to $\mathcal{O}$ ~~as~~ an equivalence.),

Ex: Let $\mathcal{O}$ be an orbifold/manifold. A proper étale groupoid presenting $\mathcal{C}$ can be obtained by:

Take ① $\mathrm{Ob}\,\mathcal{C}$ to be any manifold

② choose a smooth map $\mathrm{Ob}\,\mathcal{C} \to \mathcal{O}$ which is étale (a covering map) or surjective.

③ $\mathrm{Mor}\,\mathcal{C} = \mathrm{Ob}\,\mathcal{C} \times_{\mathcal{O}} \mathrm{Ob}\,\mathcal{C}$.

ex: $\mathbb{R}^2 / (\mathbb{Z}/3)$ can be presented as:

$\mathrm{Ob}\,\mathcal{C} = \mathbb{R}^2$

$$\mathrm{Mor}\,\mathcal{C} = \underset{\text{id}}{\mathbb{R}^2} \amalg \underset{\text{rotate by } 2\pi/3}{\mathbb{R}^2} \amalg \underset{\text{rotate by } 4\pi/3}{\mathbb{R}^2}.$$

e.g., $\{U_i\}$ cover of M

$\rightsquigarrow \mathrm{Ob}\,\mathcal{C} = \coprod U_i$ s.t.

$\mathrm{Mor}\,\mathcal{C} = \coprod_{i,j} U_i \cap U_j$.

To any orbifold $\mathcal{O}$, one can associate two natural topological spaces:

① The coarse space:

$$|\mathcal{O}| := \mathrm{Ob}\,\mathcal{C}/s(m)\sim t(m) \ \forall m \in \mathrm{Mor}\,\mathcal{C}.$$

$\mathcal{C}$ proper $\Rightarrow |\mathcal{O}|$ is Hausdorff (& locally homeo. to $\mathbb{R}^n/G$).

(warning: converse is false)

② The classifying space:

$B\mathcal{O} :=$ geometric realization of nerve of $\mathcal{C}$

$$= \left(\coprod_{k\geq 0} \mathrm{Mor}\,\mathcal{C}\ {}_s\!\times_t \cdots {}_s\!\times_t \mathrm{Mor}\,\mathcal{C} \times \Delta^k \right) \Big/ \sim .$$

There's a natural map $\underset{\text{defined up to htopy equiv.}}{B\mathcal{O}} \longrightarrow \underset{\text{defined up to homeo.}}{|\mathcal{O}|}$, and the fiber over $p \in |\mathcal{O}|$ is BG_p where G_p is the isotropy group at p.

Example: $G \curvearrowright M$, $\mathcal{O} = M/G$,

$|M/G| =$ topological quotient, and

$B(M/G) =$ homotopy quotient $(M \times EG)/G$.

Rmk: Best not to restrict to "effective orbifold" (i.e., locally modeled on $\mathbb{R}^n/G$ where $G \hookrightarrow \mathrm{Diff}(\mathbb{R}^n)$), rather than $G \xrightarrow{\text{any}} \mathrm{Diff}(\mathbb{R}^n)$),

b/c $\mathcal{M}_{1,1}$ and $\mathcal{M}_{2,0}$ aren't effective.

Rmk: Yoneda says that

$$\mathrm{Man} \hookrightarrow \mathrm{Fun}(\mathrm{Man}^{op}, \mathrm{Set})$$
$$M \longmapsto C^\infty(-, M) = h_M.$$

& h_M satisfies descent/is a sheaf.

In fact, we have another fully faithful embedding

$$\mathrm{Orb} \hookrightarrow \mathrm{Fun}(\mathrm{Man}^{op}, \mathrm{Groupoids})$$
$$\mathcal{O} \longmapsto h_{\mathcal{O}} := C^\infty(-, \mathcal{O})$$

(this is not Yoneda lemma, b/c not looking at $\mathrm{Fun}(\mathrm{Orb}, \mathrm{Gp})$).

& this functor $h_{\mathcal{O}}$ is a stack (analogue of being a sheaf. — so morphisms can be defined locally)

One can use this as a <u>definition</u> of an orbifold.

Fundamental class of an orbifold: have

$$B\mathcal{O} \longrightarrow |\mathcal{O}|.$$

In rational homology:

$H_*(B\mathcal{O};\mathbb{Q}) \xrightarrow{\cong} H_*(|\mathcal{O}|;\mathbb{Q})$ is an isomorphism b/c

$H_*(BG;\mathbb{Q}) = H_*(pt;\mathbb{Q})$. required even for $\mathbb{Z}/2$ coeffs! (e.g., S^1 mod reflection)

Suppose $\mathcal{O}$ is locally orientable (so locally modeled on $\mathbb{R}^n/G$ where G acts by orientation preserving ...)

$\Rightarrow$ get an orientation sheaf on $|\mathcal{O}|$.

Claim: $\mathcal{U} \longmapsto H_n^{BM}(\mathcal{U};\mathbb{Q})$ $n = \dim \mathcal{O}$ (for open $\mathcal{U} \subseteq |\mathcal{O}|$)

is a sheaf.

Pf: the point is just that have M-V:

$$H_{n+1}^{BM}(U\cap V) \to H_n^{BM}(U\cup V) \to H_n^{BM}(U) \oplus H_n^{BM}(V) \to H_n^{BM}(U\cap V)$$

$H_{n+1}^{BM}(U\cap V) = 0$ b/c $|\mathcal{O}|$ locally $\mathbb{R}^n/G$, which is a CW cplx. dim n.

QED.

H_*^{BM} = Borel-Moore homology

C_*^{BM} = locally finite singular chains. (cover a given cpt. set, only finitely many chains hit it)

e.g., $H_n^{BM}(\mathbb{R}^n) = \mathbb{Z}$

$H_0^{BM}(\mathbb{R}^n) = 0$.

Locally we set $[\mathbb{R}^n/G] = \frac{1}{|G|}\pi_*[\mathbb{R}^n]$.

$\Rightarrow$ sheaf property gives $[\mathcal{O}] \in H_n(|\mathcal{O}|;\mathbb{Q})$ (strictly speaking, w/ coeffs. in the orientation <u>sheaf</u>).

Question: What about lifting $[\mathcal{O}]$ to generalized homology theories?

(good reason to care about G-equiv-spectra).

- For any manifold M, have $[M] \in \pi_0^{st}(M^{-TM})$ stable cohomotopy theory

$\Rightarrow$ get a fund. class in E-homology for any E, provided we specify an iso.

$TM \wedge E \cong E$ over M.

- what about for orbifolds?