Implicit Atlases on moduli spaces of holomorphic curves

Toy case:
$$\begin{array}{ll} E & \text{vector bundle.} \\ {\scriptstyle\pi}\downarrow\uparrow{\scriptstyle s} & \\ M & \text{m'fold.} \end{array}$$

If $s \pitchfork 0$, then $s^{-1}(0)$ is a manifold and
$$[s^{-1}(0)] = \underbrace{e(E)}_{\text{Euler class, by def'n } s^*\tau_E} \cap [M]. \quad (*)$$
τ_E Thom class,

where $\tau_E \in H^{\dim E}(E, E\setminus 0)$ is the Thom class.

If $s \not\pitchfork 0$, then LHS of (*) doesn't make sense but RHS of (*) does.
It's natural to therefore declare that
$$[s^{-1}(0)]^{vir} = e(E) \cap [M] \text{ for all sections } s, \text{ not}$$
necessarily $\pitchfork 0$.

Geometry: (X, ω, J) sympl. m'fold w/ a.c. J.

Define
$$\overline{\mathcal{M}}_g(X) = \left\{ \begin{array}{l} C \text{ nodal Riemann surface} \\ u: C \to X \end{array} \;\middle|\; \begin{array}{l} \bar\partial u = 0 \\ |\mathrm{Aut}(C,u)| < \infty \end{array} \right\} \Big/ \text{iso}.$$

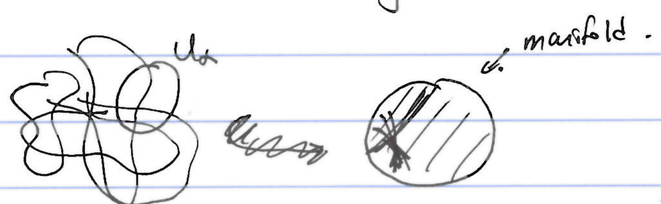
Locally, this space can be described as $\bar\partial^{-1}(0)$, where

$$\begin{array}{lc} \text{Banach bundle.} & \bigcup_{u: C \to X} W^{k-1,p}(C, \Omega^{0,1}_C \otimes_{\mathbb{C}} u^*TX) \\ & \downarrow \uparrow \bar\partial \\ \infty\text{-dim'l (Banach) m'fold} & W^{k,p}(C, X) \end{array}$$

$$\bar\partial^{-1}(0) = \{u: C \to X \mid \bar\partial u = 0\}$$

(rather, need to also vary the cplx. structure on C, enlarge the base, at least locally near C).

Want to produce charts of the following form:



$$\begin{array}{l} X = \text{"moduli space"} \\ \cup\!\!| \\ U_\alpha \hookrightarrow X_\alpha \xrightarrow{s_\alpha} E_\alpha \end{array}$$

E_α: vector space,

identifying $U_\alpha = s_\alpha^{-1}(0)$.

X_α: "thickened moduli space"

with: $\mathrm{vdim}\, X_\alpha = \mathrm{vdim}\, X + \dim E_\alpha$ & $\mathrm{vdim}\, X_\alpha = \dim X_\alpha$.

What data is required to produce such a chart?

Let A = set of all 4-tuples $(r_\alpha, D_\alpha, E_\alpha, \lambda_\alpha)$ where

1. $r_\alpha \geq 0$ is an integer.
2. $D_\alpha \subseteq V$ codim. 2 submanifold with boundary
3. E_α finite-dim'l vector space.
4. $\lambda_\alpha : E_\alpha \longrightarrow C^\infty\left(V \times \overline{\mathcal{C}}_{g,r_\alpha},\ TV \otimes_{\mathbb{C}} \Omega^{0,1}_{\overline{\mathcal{C}}_{g,r_\alpha}/\overline{\mathcal{M}}_{g,r_\alpha}}\right)$.

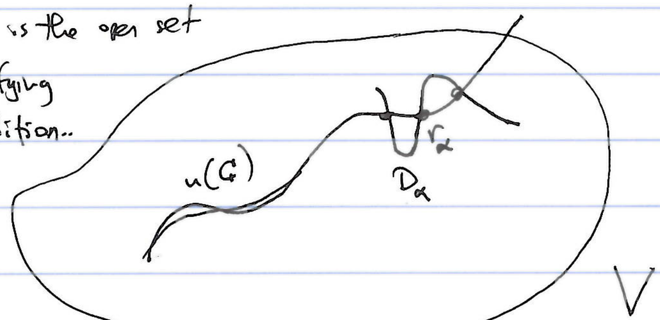
Notation: $\boxed{\begin{array}{c}\overline{\mathcal{C}}_{g,r_\alpha} \\ \downarrow \\ \overline{\mathcal{M}}_{g,r_\alpha}\end{array}}$ universal family; marked points

Define $X_\alpha = \mathcal{M}_g(V)_\alpha = \left\{ \begin{array}{l} C \\ u : C \to V \\ e_\alpha \in E_\alpha \end{array} \middle|\ \begin{array}{l} u \pitchfork D_\alpha, \\ \underbrace{|u^{-1}(D_\alpha)| = r_\alpha}, \quad \bar\partial u + \lambda_\alpha(e_\alpha, \phi_\alpha(\cdot)) = 0 \\ \Rightarrow \phi_\alpha : C \to \overline{\mathcal{C}}_{g,r_\alpha} \\ \text{(b/c get } r_\alpha \text{ marked points)} \\ + |\mathrm{Aut}(C,u)| < \infty \end{array} \right.$

$(C, u, e_\alpha) \mapsto e_\alpha \in E_\alpha$, s_α

or rather $\bar\partial u + \lambda_\alpha(e_\alpha)(u(\cdot), \phi_\alpha(\cdot)) = 0$.

& $s_\alpha^{-1}(0)$ is the open set of maps satisfying the $\pitchfork D_\alpha$ condition.. so open subset U_α of X_α.



For any finite $I \subseteq A$, have: & modulo iso. to

$$\mathcal{M}_g(V)_I = \left\{ \begin{array}{l} C \\ u : C \to V \\ (e_\alpha \in E_\alpha)_{\alpha \in I} \end{array} \middle|\ \begin{array}{l} u \pitchfork D_\alpha \quad |u^{-1}(D_\alpha)| = r_\alpha \quad \forall\, \alpha \in I \\ \bar\partial u + \sum_{\alpha \in I} \lambda_\alpha(e_\alpha)(u(\cdot), \phi_\alpha(\cdot)) = 0 \end{array} \right\}$$

Given $I \subseteq J$, note:

$$X_I \supseteq U_{IJ} = (\text{zero set}) \subseteq X_J \xrightarrow[\alpha \in J \setminus I]{\overbrace{\bigoplus s_\alpha}^{s_{J \setminus I}}} \overbrace{\bigoplus_{\alpha \in (J \setminus I)} E_\alpha}^{E_{J \setminus I}}$$

the open set of X_I where u satisfies $\pitchfork$ $\forall\, \alpha \in J$, not just I).

these structures can be axiomatized as follows.

<u>Definition</u>: X cpct, Hausdorff space

A set. Then, an <u>implicit atlas</u> on X with index set A is:

① E_α finite-dim'l vectorspace, $\alpha \in A$ ("obstruction spaces")

X_I usually non-cpct (despite including nodal curves, in applications.) ② X_I spaces, $I \subseteq A$. $(X_\emptyset = X)$ ("thickened modulispaces")

③ $s_\alpha : X_I \longrightarrow E_\alpha$, $\alpha \in I$. ("Kuranishi maps")

④ $\psi_{IJ} : (s_{J\setminus I} | X_J)^{-1}(0) \xrightarrow[\text{homeo.}]{\sim} U_{IJ} \subset X_I$ ("footprint map").

("footprint")

c.f., a combination of "footprints" & "coord. changes" appearing in traditional Kuranishi theory

⑤ $X_I^{reg} \subseteq X_I$ open ("regular locus"),

<u>satisfying</u>:

at least, they agree on the domain they're defined on.

① $\psi_{IJ}\psi_{JK} = \psi_{IK}$, $\psi_{II} = id$.

② $s_I \psi_{IJ} = s_I$ or rather $s_\alpha \circ \psi_{IJ} = s_\alpha$, $\alpha \in I \subseteq J$.

③ $U_{IJ} \cap U_{IJ'} = U_{I, J\cup J'}$

④ $\psi_{IJ}^{-1}(X_I^{reg}) \subseteq X_J^{reg}$

meaning, locally modeled on $\mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$.

☆ ⑤ $s_{J\setminus I} : X_J \longrightarrow E_{J\setminus I}$ is a <u>topological submersion</u> over $\psi_{IJ}^{-1}(X_I^{reg})$

☆ ⑥ (<u>covering axiom</u>): $X_\emptyset = \bigcup_{I \subseteq A} \psi_{\emptyset I}\left((s_I | X_I^{reg})^{-1}(0)\right)$.

Briefly, one should regard the following as the "universal" construction of an implicit atlas:

If $X = \{u \mid \bar\partial u = 0\}$, then

$$X_I = \left\{ u, (e_\alpha)_{\alpha \in I} \;\middle|\; \begin{array}{l} \bar\partial u + \sum_{\alpha \in I} e_\alpha = 0 \\ u \text{ satisfies some particular open condition,} \\ \text{depending on } \alpha \ \forall \alpha \in I \end{array} \right\}.$$

this condition helps give us "absolute coordinates" on domains of u, & hence define e_α suitably.

<u>Rmk</u>: The axioms here make no distinction between I, or J is $\emptyset$ or not $\emptyset$.

Given X with atlas A, and Y with atlas B, can define the "product atlas" on $X \times Y$ with index set $A \amalg B$ by setting

$$(X \times Y)_{I \amalg I'} := X_I \times Y_{I'} .$$

Rmk: Instead of $X \supseteq \mathcal{U}_\alpha = s_\alpha^{-1}(0) \subseteq X_\alpha \xrightarrow{s_\alpha} E_\alpha$, could consider

$$\begin{tikzcd}
s_\alpha^{-1}(0) \subseteq X_\alpha \arrow[r, "s_\alpha"] \arrow[ddd] & E_\alpha \\
\circlearrowleft & \circlearrowleft . \\
\Gamma_\alpha & \Gamma_\alpha \\
X \supseteq \mathcal{U}_\alpha = s_\alpha^{-1}(0) / \Gamma_\alpha .
\end{tikzcd}$$