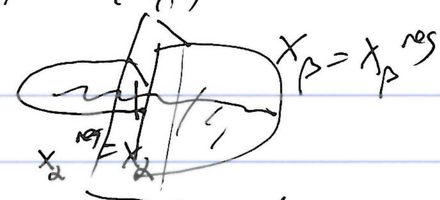
Ex: (two charts) $A = \{\alpha, \beta\}$



$\operatorname{vdim} X = 1$
$\dim X_\alpha = \dim X_\beta = 2$
$\dim X_{\alpha\beta} = 3$
$\dim E_\alpha = \dim E_\beta = 1$

J. Pardon, talk 3:

First try for def'n of $C^*_{vir}(X;A)$:

$$\begin{array}{ccc} C_{\dim X_{\alpha\beta}-*}(X_{\alpha\beta}, X_{\alpha\beta}\setminus X) & \longrightarrow & C_{\dim X_\alpha - *}(X_\alpha, X_\alpha \setminus X) \\ \wr\Vert & & \oplus \\ \check{C}^*_c(U_\alpha \cap U_\beta) & & C_{\dim X_\beta - *}(X_\beta, X_\beta \setminus X) \end{array}$$

map should be given by a cap product: difficult.

Instead: use deformation to the normal cone:

$$Y_{\alpha\beta} := \left\{ \begin{array}{c} x \in X_{\alpha\beta} \\ t \in [0,1] \\ e_\alpha \in E_\alpha \\ e_\beta \in E_\beta \end{array} \;\middle|\; \begin{array}{c} s_\alpha(x) = (1-t)e_\alpha \\ s_\beta(x) = t\, e_\beta \end{array} \right\}$$

$$(Y_{\alpha\beta})_t = \begin{cases} s_\beta^{-1}(0) \times E_\beta & t = 0 \\ X_{\alpha\beta} & 0 < t < 1 \\ s_\alpha^{-1}(0) \times E_\alpha & t = 1 \end{cases}$$

Then, take:

$$C^*_{vir}(X;A) = \left[\begin{array}{ccl} & & \check{C}^*_c(U_\alpha) \\ & & \wr\Vert \\ \text{“}\check{C}^*_\beta(U_\alpha \cap U_\beta)\text{”} & \nearrow & C_{\dim X_{\alpha\beta}-*}(X_\alpha \times E_\beta, (X_\alpha \times E_\beta)\setminus X \times 0) \\ & \searrow & \oplus \\ & & C_{\dim X_{\alpha\beta}-*}(X_{\alpha\beta}, X_{\alpha\beta}\setminus X \times [0,1]) \simeq \check{C}^*_c(U_\alpha \cap U_\beta) \\ & \nearrow & \oplus \\ \check{C}^*_c(U_\alpha \cap U_\beta) & \searrow & C_{\dim X_{\alpha\beta}-*}(X_\beta \times E_\alpha, (X_\beta \times E_\alpha)\setminus X \times 0) \\ & & \wr\Vert \\ & & \check{C}^*_c(U_\beta) \end{array} \right.$$

$\exists$ fact $\rightarrow C_{-*}(E_A, E_A \setminus 0)$

<u>Def:</u> $C^*_{vir}(X, A) := \bigoplus_{p \geq 0} \bigoplus_{I_0 \subsetneq \cdots \subsetneq I_p} C_{-p+d+\dim E_A - *}\left(\left\{ \begin{array}{l|l} e_\alpha \in E_A & t_\alpha = 0 \quad \alpha \in A \setminus I_0 \\ t \in \mathbb{R}^A_{\geq 0} & s_\alpha(x) = t_\alpha e_\alpha \quad \alpha \in A \\ x \in X^{reg}_{I_p} & \psi_{\{\alpha \in A \mid t_\alpha > 0\}, I_p}(x) \in X^{reg}_{\{\alpha \mid t_\alpha > 0\}} \\ & \text{complement of locus } e = 0 \end{array} \right\} \right)$

(weak) Thm $\simeq$ $C^*(X)$.

$\psi_{\emptyset I_p}(x) \in X$

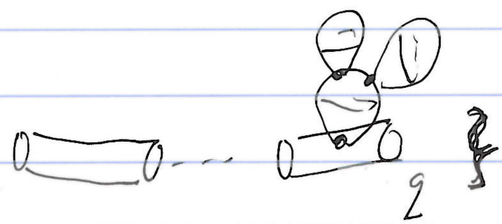
this implies a definition of a VFC: apply $C^*_{vir}(X) \longrightarrow C_{-*}(E_A, E_A \setminus 0)$!

<u>Construction of Hamiltonian Floer homology</u> M sympl. manifold.

$H: M \times S^1 \longrightarrow \mathbb{R}$

$\mathcal{P}(H) = \{\text{periodic orbits of } X_H\}$

Have $\overline{\mathcal{M}}(p,q) = \{\text{possibly broken trajectories}\}$



Declare $\overline{\mathcal{M}}(p_0, \ldots, p_k) = \overline{\mathcal{M}}(p_0, p_1) \times \cdots \times \overline{\mathcal{M}}(p_{k-1}, p_k)$

On $\overline{\mathcal{M}}(p,q)$, method from last lecture gives atlas $A(p,q)$. In fact, it makes sense to define thickened versions of

$\overline{\mathcal{M}}(p_0, \ldots, p_k)$ for any subset of $\bar{A}(p_0, \ldots, p_k) = \coprod_{i<j} A(p_i, p_j)$.

(*) [<u>Goal</u>(*): define <u>chains</u> $[\overline{\mathcal{M}}(p,q)]^{vir}$ such that

$$\partial [\overline{\mathcal{M}}(p,q)]^{vir} = \sum_r [\overline{\mathcal{M}}(p,r)]^{vir} \times [\overline{\mathcal{M}}(r,q)]^{vir} \quad].$$

A <u>thickening datum</u> $\alpha \in A(p_i, p_j)$ only "applies" over the part of the trajectory from p_i to p_j.

has more thickening data

there's an inclusion of atlases this way

Have relationships $\bar{A}(p_0,\dots,p_r) \longleftrightarrow \bar{A}(p_0,\dots,\hat{p}_i,\dots,p_k)$

$$\bar{\mathcal{M}}(p_0,\dots,p_k) \hookrightarrow \bar{\mathcal{M}}(p_0,\dots,\hat{p}_i,\dots,p_k)$$

trajectories which may be broken at <u>more</u> than $p_0,\dots,p_k$.

& similarly,

$$\bar{A}(p_0,\dots,p_i) \sqcup \bar{A}(p_i,\dots,p_k) \hookrightarrow \bar{A}(p_0,\dots,p_k)$$

$$\bar{\mathcal{M}}(p_0,\dots,p_i) \times \bar{\mathcal{M}}(p_i,\dots,p_k) \cong \bar{\mathcal{M}}(p_1,\dots,p_k)$$

Let $\mathcal{S} = \left\{ \underbrace{(p_0,\dots,p_k)}_{\text{ignore now}} \;\middle|\; \begin{array}{l} k \geq 1 \\ p_i \in \mathcal{P}(H) \end{array} \right\}$. $\mathcal{S}$ is a category in which morphisms

& homology class of cylinders $p_{i-1} \to p_i$

are given by forgetting some interior p_i $(0 < i < k)$.

(& $\mathcal{S}$ is vaguely monoidal: can concatenate $(p_0,\dots,p_i)$ $(p_i,\dots,p_k)$ to $(p_0,\dots,p_k)$.)

<u>Def:</u> An $\mathcal{S}$-module valued in $\mathcal{C}^{\otimes}$ is:

1. A functor $X : \mathcal{S} \to \mathcal{C}$
2. Maps $X(p_0,\dots,p_i) \times X(p_i,\dots,p_k) \to X(p_0,\dots,p_k)$
3. Compatibility relations ...

<u>Example:</u> $(p_0,\dots,p_k) \longmapsto \bar{\mathcal{M}}(p_0,\dots,p_k)$ is an $\mathcal{S}$-module.

<u>Example:</u> $(p_0,\dots,p_k) \longmapsto C^{v*+\mathrm{vdim}(p_0,\dots,p_k)}\left(\bar{\mathcal{M}}(p_0,\dots,p_k) \text{ rel } \partial\right)$

is an $\mathcal{S}$ module.

(codim 1 pushforwards are coboundary maps)

Write T for a general element of $\mathcal{S}$.

Can write

$$\check{C}^{*+\mathrm{vdim}\,\overline{\mathcal{M}}(T)}(\overline{\mathcal{M}}(T)\ \mathrm{rel}\ \partial) = \Big[\check{C}^*(\mathcal{M}(T)) \to \bigoplus_{\mathrm{codim}(T'/T)=1} \check{C}^*(\overline{\mathcal{M}}(T')) \longrightarrow \bigoplus_{\mathrm{codim}=2} \check{C}^*(\overline{\mathcal{M}}(T')) \to \cdots$$

& now the coboundary maps on chain level are <u>clear</u>..

<u>Example</u>: $\mathbb{Z}$ denotes the <u>constant</u> $\mathcal{S}$-module.

<u>Lemma</u>: A map of $\mathcal{S}$-modules from

$$\check{C}^{*+\mathrm{vdim}\,\overline{\mathcal{M}}(T)}(\overline{\mathcal{M}}(T)\ \mathrm{rel}\ \partial) \longrightarrow \mathbb{Z}$$

is exactly the same as a collection of chains as in <u>Goal (*)</u>.

write

$$\mathbb{Z}[\mathcal{S}_{/T}] := [\mathbb{Z} \to \bigoplus_{\mathrm{codim}(T'/T)=1} \mathbb{Z} \to \text{---}]$$

<u>Lem</u>: a map involving $\text{--} \to \mathbb{Z}$ is a collection of #'s "counts" satisfying ---.

Functoriality of $C^*_{vir}(X;A)$ gives a diagram of $\mathcal{S}$-modules:

$$\begin{array}{c} C^*_{vir}(\overline{\mathcal{M}}(T)\ \mathrm{rel}\ \partial) \longrightarrow C_*(E_A, E_A \setminus 0) \simeq \mathbb{Z} \\ \text{\rotatebox{90}{$\cong$}} \\ \check{C}^*(\overline{\mathcal{M}}(T)\ \mathrm{rel}\ \partial). \end{array}$$

$\mathcal{S}$-modules valued in spectra should be useful for Floer homotopy. Namely a map of $\mathcal{S}$-modules (what should be a map in the ∞-category sense now)

$$\left(T \longmapsto \begin{array}{c}\text{complex built out of} \\ \mathbb{S} \text{ for every} \\ T' \in \mathcal{S}_{/T}\end{array}\right) \longrightarrow (T \to \mathbb{S})$$

is exactly the data needed to define a Cohen-Jones-Segal Floer homotopy type.