E. Riehl I. Aim: define <u>homotopy coherent diagram</u>

§ <u>Motivation</u> : $X$ $G$-space, $Y \xrightarrow[\sim]{} X$, is $Y$ a $G$-space?

htpy equiv

Need: $h, g \in G$

If:

$$\begin{tikzcd} Y \arrow[r, "f^{-1}"] \arrow[d, dashed, "g_*"] & X \arrow[d, "g_*"] \\ Y & X \arrow[l, "f"'] \end{tikzcd}$$

(specify $f, f^{-1}$)

then

$$h_* g_* = f h_* f^{-1} f g_* f^{-1}$$
$$(hg)_* = f (hg)_* f^{-1} = f h_* g_* f^{-1}.$$

These are <u>homotopic</u> $f^{-1} f \simeq \mathrm{id}_X$

A $G$-space $X$ is a diagram (e.g., a functor)

$$BG \xrightarrow{X} \text{Spaces}.$$

Note Spaces (the category) has mapping spaces*: Means if $X, Y \in$ Spaces, then $\mathrm{Map}(X,Y)$ ~~defn~~ is a "space" (modulo pt.-set topology issues, deal w/ later) whose points are $f : X \to Y$ and whose paths are <u>homotopies</u>.

$f, g : X \to Y$ are <u>homotopic</u> iff they are in the same $\pi_0 \mathrm{Map}(X,Y)$.

An up to homotopy $G$-space $Y$ is a diagram

$$BG \longrightarrow h\text{Spaces} \begin{cases} \text{Spaces} \\ \text{htpy classes of maps} \end{cases}.$$

<u>defn</u>: A diagram is a functor $\mathcal{A}$ (small category) $\longrightarrow$ Spaces, & a <u>htpy-commutative diagram</u> is also $\cdots$ $\mathcal{A} \longrightarrow h\,\text{Space}$.

Q: Is every htpy commutative diagram "realized" by a commutative diagram? (in the sense of ex-, prev def'n in lecture notes).

**Thm:** [Dwyer-Kan-Smith] A htpy commutative diagram may be realized by a strictly commutative diagram <u>if and only if</u> it may be extended to a <u>homotopy coherent diagram</u>.

(in fact, the thm is stronger: equiv. of moduli space of rectifications & htpy coherent structures)

So, in particular in our example $Y : BG \longrightarrow h\,\mathcal{S}paces$ may be made <u>homotopy coherent</u>.

## § Shape of htpy coherence

Consider $\omega = 0 \to 1 \to 2 \to 3 \to \cdots$

A htpy commutative diagram $\omega \to h\,\mathcal{S}pace$ has

- spaces $X_j$ $\forall j \in \omega$.
- functions $f_{jk} : X_j \to X_k$ $\forall j < k \in \omega$. s.t.

  $f_{ik} \simeq f_{jk} \circ f_{ij}$ wherever $i<j<k$.

to make htpy coherent:

- pick homotopies $h_{ijk}$ from $f_{ik}$ ~~to~~ $\to f_{jk} \circ f_{ij}$ $\quad i<j<k$ (paths in $Map(X_i, X_k)$).

- For $i<j<k<l$ $\qquad Map(X_i, X_l)$

$$\begin{tikzcd} f_{il} \arrow[r, "h_{ikl}"] \arrow[d, "h_{ijl}"'] \arrow[dr, "\text{another htpy here.}"] & f_{kl} \circ f_{ik} \arrow[d, "f_{kl} \circ h_{ijk}"] \\ f_{jl} f_{ij} \arrow[r, "h_{jkl} \circ f_{ij}"'] & f_{kl} f_{jk} f_{ij} \end{tikzcd}$$

2-simplex

2-simplex

need: a <u>2-homotopy</u> filling this square $h_{ijkl}$.

- For $i<j<k<l<m$, in $Map(X_i, X_m)$, need to pick a <u>3 homotopy</u> filling a cube.

etc.

Simplicial sets are a model for spaces.

Def'n: A simplicial category $A_\bullet$ consists of

- categories $A_n$ for $n \geq 0$ w/ ob $A_n$ = ob $A$

A map in $A_n$ is called an n-arrow, along with

$$A_0 \underset{\text{target}}{\overset{\text{source}}{\leftleftarrows\!\!\rightarrow}} A_1 \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; A_2 \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; A_3 \; \cdots$$

(a simplicial object in cat of cats w/ ob $\mathcal{A}$ as objects & all functors identity on objects).

Prop: TFAE:

- A simplicial category $A_\bullet$ ob $\mathcal{A}$ as objects.
- A category enriched over simplicial sets ...

Proof: If $x, y \in$ ob $\mathcal{A}$, an n-arrow $x \to y$ is an n-simplex in $\mathcal{A}(x,y)$.

Def'n (free resolution): For a cat $\mathcal{A}$, $\mathcal{C}\mathcal{A}$ is a simplicial category, with ob $\mathcal{C}\mathcal{A}$ = ob $\mathcal{A}$. Write $U\mathcal{A}$ = underlying graph of $\mathcal{A}$.

$FU\mathcal{A}$ is the free category on the underlying category $U\mathcal{A}$.

$$(\mathcal{C}\mathcal{A})_n = (FU)^{n+1}\mathcal{A}.$$

(adjunction F & U?
& all arrows are unit & counit of adjunction!)

$$FU\mathcal{A} \leftrightarrows (FU)^2\mathcal{A} \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; (FU)^3\mathcal{A}$$

duplication or deletion of parentheses.

Exercise: compute $\mathcal{C}(\omega)$. (these will be "atomic n-arrows" of relevance to Steph!)

A 0-arrow $x \to y$ is:

$$x \to a_1 \to a_2 \to a_3 \to \cdots a_{n+1} \to y, \text{ e.g.,}$$

a sequence of composable arrows. An n-arrow is a sequence of composable arrows – each inside exactly n-parentheses.

$(FU)^2\mathcal{A}$ is chains $(a \to a_1 \to \cdots)(- -)(- - -)(- -)$

Ex: $BG$, there's a simplicial category $\mathfrak{C}(BG)$

$\mathrm{ob}\,\mathfrak{C}BG = *$, 1-arrows $FUBG = FG$.

$$\begin{tikzcd} FUBG & FU^2BG \arrow[l, shift right, "\text{multiply adjacent chains}"'] \arrow[l, shift left, "\text{forget parentheses}"] \end{tikzcd}$$

$(g's\text{---})(g's\text{---})(g's\text{---})$

Def: A $\underline{\text{htpy coherent diagram}}$ of shape $A$ is a simplicial functor $\mathfrak{C}A \to \text{Spaces}$.

$\underline{\text{Ex}}$: There is a composition map $\mathfrak{C}A \xrightarrow{\varepsilon} A$ "counit of an adjunction".

compose all arrows.

(a local htpy equivalence, e.g., between any two objects).

So, any commutative diagram gives a htpy commutative one

$\underline{\text{Def'n}}$: A htpy coh. natural transformation between htpy coh. diagrams $F, G : A \to \text{Spaces}$, written $\alpha : F \to G$ is

$$\begin{tikzcd} \mathfrak{C}(A) \arrow[d, "0"] \arrow[dr, "F"] & \\ \mathfrak{C}(A \times [1]) \arrow[r, "\alpha"] & \text{Spaces} \\ \mathfrak{C}(A) \arrow[u, "1"'] \arrow[ur, "G"'] & \end{tikzcd}$$

(note may not be invertible, unlike homotopies)

$[1] = 0 \to 1$.

Note: Given $\alpha : F \to G$, $\beta : G \to H$ htpy coherent natural transformations, these do $\underline{\text{not}}$ compose uniquely. A composite is witnessed by a homotopy coherent diagram of shape $A \times [2]$, where $[n] = 0 \to 1 \to 2 \to \cdots \to n$.

$$\begin{tikzcd} & G \arrow[dr, "\beta"] \arrow[d, phantom, "\Downarrow"] & \\ F \arrow[ur, "\alpha"] \arrow[rr, "\beta\alpha"'] & {} & H \end{tikzcd}$$

Def'n: $\mathrm{Coh}(A, \text{Spaces})$ is a simplicial set whose $n$-simplices are $\mathfrak{C}[A \times [n]] \to \text{Space}$.

**Thm:** (Boardman-Vogt): Since the mapping spaces in Space are "Kan complexes," $\mathrm{Coh}(A, \text{Space})$ is a quasi-category.

Kan complex:

k-horn (remove int, & $k^{th}$ face)
$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & \mathrm{Map}(X,Y) \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$
$0 \le k \le n$

$\mathrm{Coh}(A, \text{Space})$ is a quasi-cat: if
$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & \mathrm{Coh}(A, \text{Space}) \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$
$\exists$ when $k \neq 0, n$

$0 < k < n$.

Thm (Vogt, Cordier-Porter): The natural map
$$\text{Space}^A \to \mathrm{Coh}(A, \text{Space})$$ induces an equivalence of

of categories
$$\mathrm{Ho}(\text{Spaces}^A) \xrightarrow{\simeq} \mathrm{Ho}\,\mathrm{Coh}(A, \text{Spaces}).$$

$$\left( \begin{tikzcd} \mathrm{Cat} \arrow[r, shift left, "F"] & \mathrm{DirGraph} \arrow[l, shift left, "U"] \end{tikzcd} \right), \quad \eta : \mathrm{id}_{\mathrm{Graph}} \to UF, \quad \varepsilon : FU \to \mathrm{id}_{\mathrm{Cat}}$$

Rmk: defining here "strictly unital" diagrams.

$\exists$ a more general notion w/ different notions of identities, & it's equivalent.