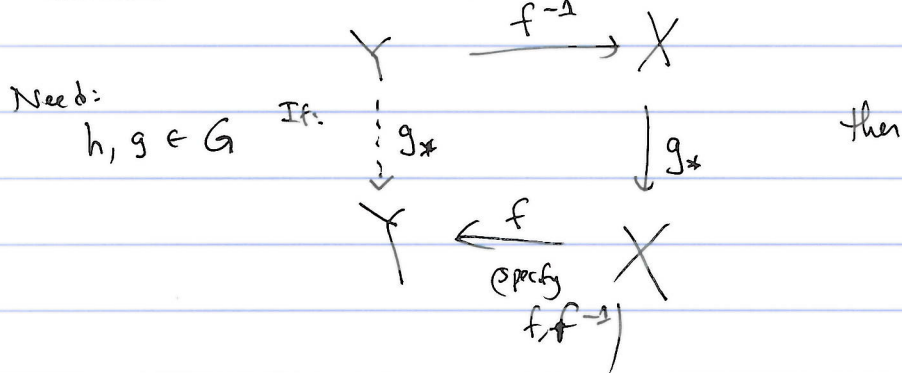


E. Pichl I. Ann: define homotopy sheet diagram

≅-Motivation: X G -space, $Y \xrightarrow{\sim} X$, is Y a G -space? ↓ htpy equiv



$$h_* g_* = f_* h_* f_*^{-1} g_* f_*^{-1}$$

$$(hg)_* = f_* (hg)_* f_*^{-1} = f_* h_* g_* f_*^{-1} .$$

These are homotopic
 $f_*^{-1} f_* \sim id_X$

A G -space X is a diagram (e.g., a functor)

$$BG \xrightarrow{X} \text{Spaces} .$$

↓ htpy classes

Note Spaces has mapping spaces^{*}: Means if $X, Y \in \text{Spaces}$, then $\text{Map}(X, Y)$ ~~is~~ ^{is} a "space" (modulo pt.-set topology issues, deal w/ later)

whose points are $f: X \rightarrow Y$ and whose paths are homotopies.

$f, g: X \rightarrow Y$ are homotopic iff they are in the same $\pi_0 \text{Map}(X, Y)$.

An up to homotopy G -space Y is a diagram,

$$BG \longrightarrow h \text{ Spaces} \begin{cases} \text{Spaces} \\ \text{htpy classes of maps} \end{cases} .$$

def'n: A diagram is a functor \mathcal{A} (small category) $\rightarrow \text{Spaces}$, & a htpy-commutative diagram is also $\mathcal{A} \rightarrow h \text{ Space}$.

Q: Is every htpy commutative diagram "realized" by a commutative diagram? (in the sense of ex., prec def'n in lecture notes).

Thm: [Dwyer-Kan-Smith] A htpy commutative diagram may be realized by a strictly commutative diagram if and only if it may be extended to a homotopy coherent diagram.

(in fact, the theorem is stronger: equiv. of moduli space of retracts & htpy coherent structures)
 in particular in our example
 So, $Y : BG \rightarrow h\text{ Spaces}$ may be made homotopy coherent.

§ Shape of htpy coherence

Consider $\omega = 0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow \dots$

A htpy commutative diagram $\omega \rightarrow h\text{ Space}$ has

- spaces $X_j \forall j \in \mathbb{N}$.

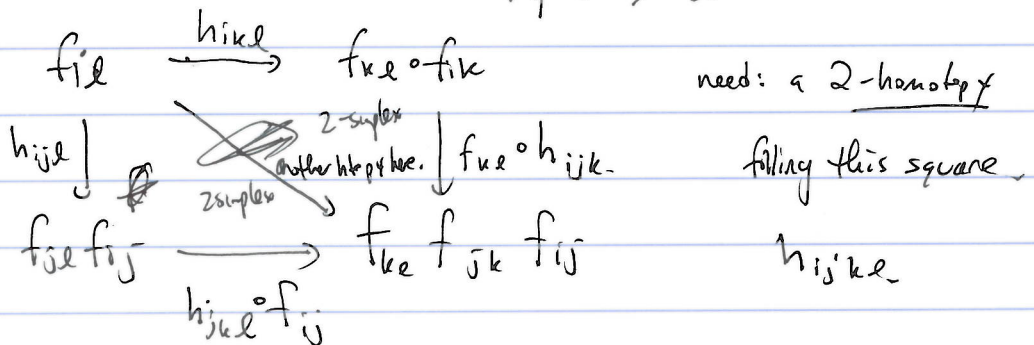
- functions $f_{jk} : X_j \rightarrow X_k \forall j < k \in \omega$. s.t.

$$f_{ik} \simeq f_{ik} \circ f_{ij} \text{ whenever } i < j < k$$

to make htpy coherent, :

- pick homotopies h_{ijk} from $f_{ik} \rightarrow f_{ik} \circ f_{ij} \quad i < j < k$
 (paths in $\text{Map}(X_i, X_k)$)

- For $i < j < k < l$ $\text{Map}(X_i, X_l)$



need: a 2-homotopy filling this square $h_{ijl, k}$

- for $i < l < k < l = m$, in $\text{Map}(X_i, X_m)$, need to pick a 3-homotopy filling a cube.

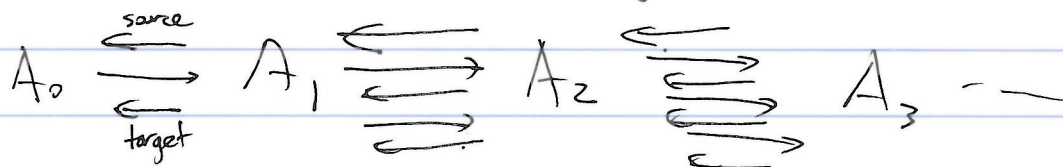
etc.

Simplicial sets are a model for spaces.

Def'n: A simplicial category \mathcal{A}_\bullet consists of

- categories A_n for $n \geq 0$ w/ $ob A_n = ob A$

A map in A_n is called an n-arrow, along with



(a simplicial object is a seq of obj w/ ob of A_n objects & all factors identity on objects)

Prop: TFAE:

- A simplicial category A_\bullet w/ $ob A_n = ob A$
- A category enriched over simplicial sets

Proof: If $x, y \in ob A$, an n -arrow $x \rightarrow y$ is an n -simplex in $\mathcal{A}(x, y)$.

Def'n (free resolution): For a cat A , $\mathcal{C}A$ is a simplicial category, with

$ob \mathcal{C}A = ob A$. Write $UA =$ underlying graph of A .

FUA is the free category on the underlying category UA .

$$(\mathcal{C}A)_n = (FU)^{n+1}A$$

(a structure FBU ?
& all arrows as unit & counit of adjunction)

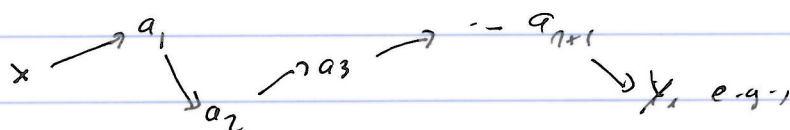
$$FUA \rightleftharpoons (FU)^2A \rightleftharpoons (FU)^3A$$

Exercise: compute $\mathcal{C}(w)$. (there will be "atomic n -arrows" of relevance to people!)

duplicate of delete of parentheses

$(FU)^2A$ is draws $(a \rightarrow a_1 \rightarrow \dots) (\dots) (\dots) (\dots)$

A 0-arrow $x \rightarrow y$ is:

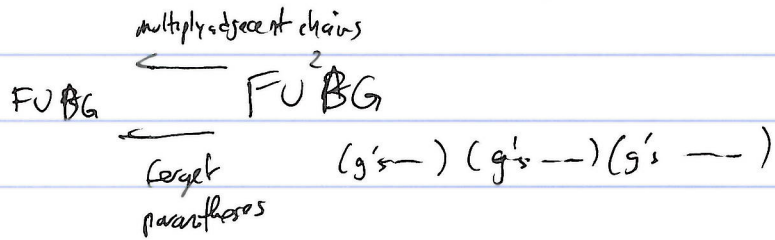


a sequence of composable arrows. An n -arrow is a sequence of composable arrows - each inside exactly n -parentheses.

Ex: BG , there's a simplicial category

$\mathcal{C}(BG)$

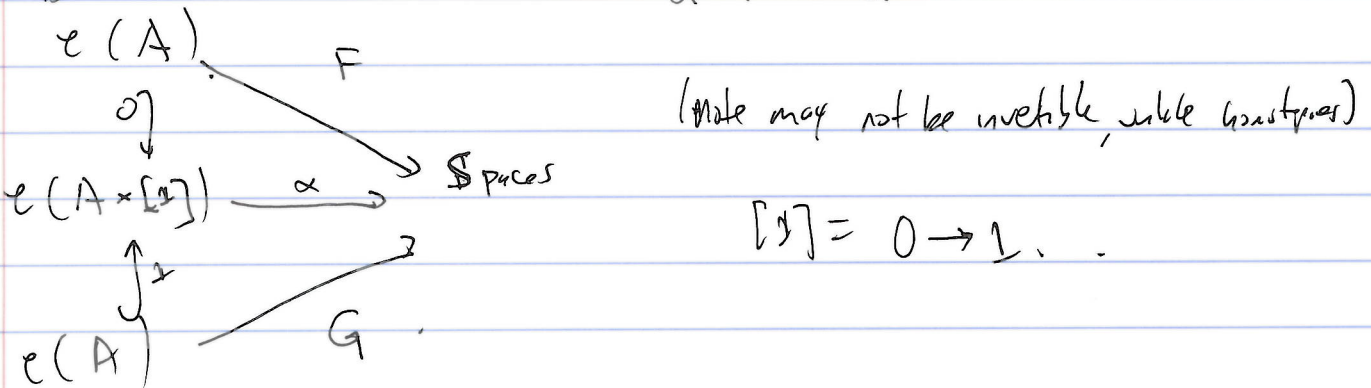
ob $\mathcal{C}(BG) = *$, 1-arrows $FU BG = FG$.



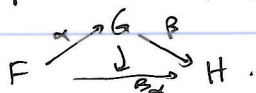
Def: A htpy coherent diagram of shape A is a simplicial functor $\mathcal{C}A \rightarrow \text{Spaces}$.

Ex: There is a composition map $\mathcal{C}A \xrightarrow{\varepsilon} A$ "count of a collocation"
 (a local htpy equivalence, e.g., between any two objects) \uparrow compose all arrows.

So, any commutative diagram gives a htpy commutative one
 Def'n: A htpy coh. natural transformation between htpy coh. diagrams $F, G: A \rightarrow \text{Spaces}$, written
 is $\alpha: F \rightarrow G$

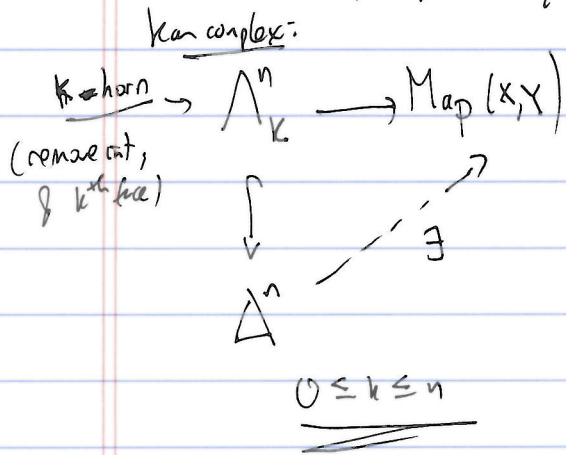


Note: Given $\alpha: F \rightarrow G, \beta: G \rightarrow H$ htpy coherent natural transformations, those do not compose uniquely. A composite is witnessed by a htpy coherent diagram of shape $A \times [2]$, where $[n] = 0 \rightarrow 1 \rightarrow 2 \rightarrow \dots \rightarrow n$.

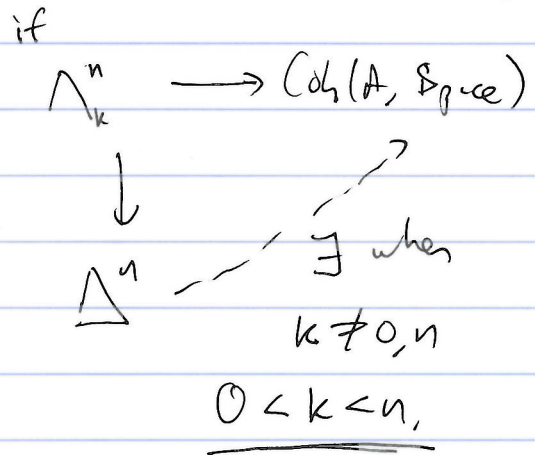


Def'n: $\text{Coh}(A, \text{Space})$ is a simplicial set whose n -simplices are $\cdot \in \mathbb{C}(A \times [n]) \rightarrow \text{Space}$.

Thm: (Boardman-Vogt) • Since the mapping spaces in Space are "Kan complexes," $\text{Coh}(A, \text{Space})$ is a quasi-category.



$\text{Coh}(A, \text{Space})$ is a quasi-cat:



Thm (Vogt, Carter-Porter): The natural map $\text{Space}^A \rightarrow \text{Coh}(A, \text{Space})$ induces an equivalence of categories

$$\text{Ho}(\text{Spaces}^A) \xrightarrow{\cong} \text{Ho}(\text{Coh}(A, \text{Spaces})).$$

$$\left(\text{Cat} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{U} \end{array} \text{Dir Graph} \right), \quad \begin{array}{l} \eta: \text{id}_{\text{Graph}} \rightarrow UF \\ \varepsilon: FU \rightarrow \text{id}_{\text{Cat}} \end{array}$$

Remark: defining the "strictly unital" diagrams.

\mathcal{I} = more general notion of different notions of identities, & its equivalent