

Eriehl II

Aim: diagrams in a q.cat. are automatically homotopy coherent. → reconceptualize Cat.

Last time: a homotopy coherent diagram of shape \mathbb{S} is

$Cat \rightarrow \text{Space}$, e.g. $a \in A \rightsquigarrow X_a \in \text{Space}$

$Cat(a,b) \rightarrow \text{Map}(X_a, X_b)$

Set (in a transply, paths are "invertible!"),

Today: Let \mathbb{S} be any category encoded in spaces Kan complexes. "locally Kan categories!"

§. Simplicial Computed

$f: x \rightarrow y$ is atomic if it can't be factored (f isn't id_x).

A category is freely generated (by a reflexive directed graph of atomic arrows) iff every arrow admits a unique factor uniquely into atomics.

Defn (simplicial computed): A simplicial category $A_\bullet = (A_0 \rightrightarrows A_1 \rightrightarrows A_2 \dashv \vdash \dots \text{ all } v \text{ of } \partial A)$

is a simplicial computed if

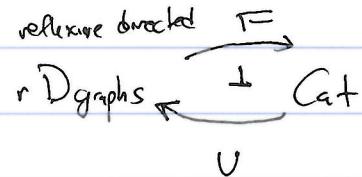
(i) each A_n is freely generated (by "atomic-n-arrows")

(ii) For each $[n] \rightarrow [m]$ in Δ the functor $A_n \rightarrow A_m$ preserves atomics.

Prop: $\mathbb{S}Cat$ is a simplicial computed.

Proof: $Cat_\bullet =$

$FU A \xleftarrow{\epsilon_{FU}} (FU)^2 A \xleftarrow{\epsilon_{FU^2}} (FU)^3 A \dashv \vdash \dots$



- 0-arrows are strings of composable morphisms

- atomic 0-arrows are single arrows.

- 1-arrows are " " in 1 set of parentheses.

$$(fgh)(k)(l)(mn)$$

- Atomic 2-arrows have form (-string).

- n-arrows - - - n-parenthesized.

- atomic = one outer parenthesis,

8 (i) ← (ii) doubling up on parentheses preserves property of having a single parenthesis on outside.

Ex: $\omega = 0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow - -$

by prop, $\mathbb{C}\omega$ is a simplicial complex, with

- $\text{ob } \mathbb{C}\omega = \text{ob } \omega = \{n \geq 0\}$

- 0-arrow from j to k , $j \leq k$ (only interesting case) is a union of composable arrows, determined by a subset T :

$$\{j, k\} \subset T \subset [j, k] = \{t \in \omega \mid j \leq t \leq k\}$$

- 1-arrow " " " " in brackets, given by

$$\{j, k\} \subset T^1 \subset T^0 \subset [j, k]$$

\uparrow bracketing of \nwarrow string of arrows
 parentheses.

- n -arrow is an unbracketed sequence of composable arrows

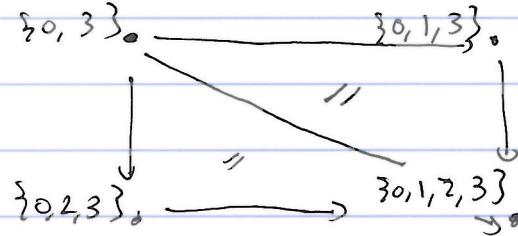
$$\{j, k\} \subset T^n \subset \dots \subset T^2 \subset T^1 \subset T^0 \subset [j, k]$$

\uparrow \nwarrow underlying sequence

nested seq. (b/c parentheses need to be "well-formed")

In total, $\mathbb{C}\omega(j, k)$ has 2^{k-j-1} vertices.

E.g., $\mathbb{C}\omega(0, 3)$



Upshot $\mathbb{C}\omega(j, k) = \left\{ \begin{array}{ll} \emptyset & k < j \\ * & j = k \\ (\Delta)^{k-j-1} & k > j \end{array} \right. . (*)$

Next, an atomic 0-arrow is the case $T^0 = \{j, k\}$

" 1-arrow

$$T^1 = \{j, k\}$$

n -arrow

$$T^n = \{j, k\} \quad (\text{location of outer parentheses})$$

e.g., atomic n -arrows contain the initial vertex in the cube. (*)

S. Homotopy coherent realization and nerve functors

Inside ω , have $[n] = 0 \rightarrow 1 \rightarrow \dots \rightarrow n$ full subcategory.

Then, $\mathcal{C}[n]$ is the homotopy coherent n-simplex.

We have a functor $\Delta \xrightarrow{\mathcal{C}} s\text{Cat}$

$$[n] \longmapsto \mathcal{C}[n].$$

Def'n: \mathcal{C} is left adjoint to N .

$$s\text{Set} \begin{array}{c} \xrightarrow{\mathcal{C}} \\ \perp \\ \xleftarrow{N} \end{array} s\text{Cat}_{\leq g}$$

Homotopy coherent nerve $(N\mathbb{S})_n := \{ \mathcal{C}[n] \rightarrow \mathbb{S} \}$

"homotopy coherent diagrams
of shape \mathbb{S} "

$$(\text{recall } \text{Ob}(A, \mathbb{S})_n = \{ \mathcal{C}(A \times [n]) \rightarrow \mathbb{S} \}) \quad N\mathbb{S} = \text{Colim}(\mathbb{1}, \mathbb{S}). \quad \text{"coherent diagrams in the shape of a object"}$$

Homotopy coherent realization (not a std. name in the literature)

\mathcal{C} is left Kan extension / Yoneda embedding $[n] \mapsto \Delta_n$.

$$\Delta \xleftarrow{Y} s\text{Set} \quad \text{meaning: } \mathcal{C}\Delta^n := \mathcal{C}[n]$$

$$\mathcal{C}X := \text{colim } \mathcal{C}[n]^{\circ} \quad (\text{being explicit here}),$$

Similarly $\Delta \hookrightarrow \text{Cat}$ yields an adjunction

$$[n] \longmapsto [n] = 0 \rightarrow 1 \rightarrow \dots \rightarrow n$$

$$s\text{Set} \begin{array}{c} \xrightarrow{H\circ} \\ \perp \\ \xleftarrow{\text{"nerve"}} \end{array} \text{Cat} \quad \text{nerve of } A \text{ has } \{ [n] \rightarrow A \} \text{ as its } n\text{-simplices.}$$

$$a_0 \rightarrow a_1 \rightarrow \dots \rightarrow a_n.$$

Rmk: Last time, defined $BG \subset \text{Cat}$, the usual $s\text{Set}$ BG is nerve of BG as we defined it.

Thm: $\mathcal{C}A = CA$. That is, for any category A , the free resolution is isomorphic to homotopy coherent realization of the nerve.

Rmk: doesn't just follow from care of $[n]$ by "taking colims" as I think they do, it requires some effort.

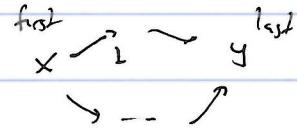
Use:

Prop: (Dugger-Spira): for any $X \in \text{Set}$, $\mathbb{C}X$ is also a simplicial category, with

$$= \text{ob } \mathbb{C}X = X_0$$

= atomic 0-arrows are 1-simplices $x \xrightarrow{f} y$

= atomic 1-arrows are non-degenerate k-simplices



(**)

Moreover

$X \hookrightarrow Y$ sSet inclusion

$\mathbb{C}X \hookrightarrow \mathbb{C}Y$ inclusion

of simplicial categories.

atomic n-arrows are k-simplices $x \rightarrow y$ $k > n$, along with

(*) $\{0, k\} = T^n \subsetneq T^{n-1} \subsetneq \dots \subsetneq T^1 \subsetneq T^0 = \{0, k\}$, i.e., an atomic n-arrow in $\mathbb{C}\Delta^k(0, k)$ that's not in any face.

[sketch]:

proof of main thm: Both $\mathbb{C}A$ and $\mathbb{C}A$ have $\text{ob } A$ as objects (in case of free resolution, by def'n, in other case of coherent realization by above)

• atomic 0-arrows are morphisms in $A = 1$ -simplices in nerve.

• atomic 1-arrows are k -composable morphisms (once articulated) in $A = a$ k -simplex in nerve,

• atomic n-arrows " " " " + n-brackets = a k -simplex plus (*) -

(plus check face + degeneracies).

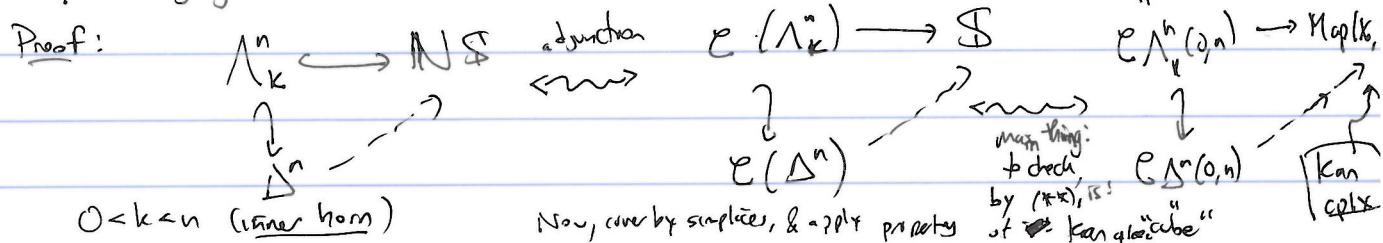
Cor: A homotopy coherent diagram of shape A in \mathbb{S} is

(i) $\mathbb{C}A \rightarrow \mathbb{S}$ OR $A \rightarrow N\mathbb{S}$ (b/c $\mathbb{C} \rightarrow N$).
simplicial functor. simplicial map (implicitly "nerve of A ").

(ii) $\text{Col}(A, \mathbb{S}) \cong (N\mathbb{S})^A$ (e.g., take internal hom in sSet from A to $N\mathbb{S}$)

Thm: (Cordier-Paré); If \mathbb{S} is a Kan complex enriched simplicial category, then $N\mathbb{S}$ is a "cubical hom" quasi-category.

Proof:



$0 < k < n$ (inner hom)

Now, cover by simplices & apply Kan complex properties:

Thm: (Boardman - Vogt):

$\text{Coh}(A, \mathbb{S})$ is a quasi-cat. if \mathbb{S} is Kan complex enriched.

Proof: $\text{Coh}(A, \mathbb{S}) \cong \text{N} \mathbb{S}^A$ + quasi-cats. are exponential ideal
(means property of being a quasi-cat. is inherited by mapping in).

To explain slogan: ^{given} \mathbb{S} cat-enriched in Kan complexes, $\& X$, then simplicial maps-

$$\begin{array}{ccc} \text{sSet} & \xrightarrow{\quad} & X \longrightarrow \text{N} \mathbb{S} \\ & \downarrow & \\ CX & \longrightarrow & \mathbb{S} \end{array}$$

homotopy coherent diagram of shape $-X$.

& remark: every qcat is equivalent to a q-cat of the form $\text{N} \mathbb{S}$.

\Rightarrow every map of simplicial sets is abstractly "homotopy coherent."

$$\begin{array}{ccc} \text{sSet} & \xrightleftharpoons[\text{Sing}]{{\sim}^{-1}} & \text{Top} \quad \text{Sing}(Y) \text{ is a Kan complex.} \end{array}$$

$\Delta^1 = 0 \rightarrow 1$ not a Kan complex
Instead, " S^∞ " or " $\mathbb{I}^\infty = 0 \overset{\sim}{\hookrightarrow} 1$ " $\& 0 \overset{\sim}{\hookrightarrow} \vee_0 \overset{\sim}{\hookrightarrow} 1$
 $\&$ higher.