www.math.jhu.edu/~eriehl/coherent-structures.pdf

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Aim: diagrams in a q.cat. are automatically htpy coherent. — reconceptualize $\mathfrak{C}\mathcal{A}$.

Last time: a homotopy coherent diagram of shape $\mathcal{A}$ is

$$\mathfrak{C}\mathcal{A} \longrightarrow \mathcal{S}pace, \text{ i.e. e.g. } a \in A \rightsquigarrow X_a \in \mathcal{S}pace$$

$$\mathfrak{C}A(a,b) \longrightarrow Map(X_a, X_b) \quad {}_s\mathcal{S}et$$

(in a Kan cplx, paths are "invertible"),

Today: Let $\mathcal{S}$ be any category enriched in ~~spaces~~ Kan complexes. "locally Kan categories"

§. Simplicial Computads

$f: x \longrightarrow y$ is atomic if it can't be factored (g isn't id_x).

A category is freely generated (by a (reflexive directed) graph of atomic arrows) iff every arrow ~~admits a unique~~ factors uniquely into atomics.

Defn (simplicial computad): A simplicial category $A_\bullet = (A_0 \rightleftarrows A_1 \rightleftarrows A_2 \;\cdots$ all w/ ob A) is a simplicial computad if

(i) each A_n is freely generated. (by "atomic n-arrows")

(ii) For each $[n] \twoheadrightarrow [m]$ $(m \geq n)$ in Δ the associated functor $A_m \longrightarrow A_n$ preserves atomics.

$$\begin{tikzcd} rDgraphs \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & Cat \arrow[l, bend left, "U"] \end{tikzcd}$$

reflexive directed

Prop: $\mathfrak{C}\mathcal{A}$ is a simplicial computad.

proof: $\mathfrak{C}\mathcal{A}_\bullet =$

$$FUA \underset{FU\varepsilon}{\overset{\varepsilon FU}{\leftleftarrows}} (FU)^2 A \leftleftarrows (FU)^3 A \;\cdots$$

- 0 arrows are strings of composible morphisms
- atomic 0-arrows are single arrows.
- 1-arrows are " " in 1 set of parentheses.

$(fgh)(k)(\ell)(mn)$

- Atomic 1-arrows have form (string).
- n arrows — — n-parenthesized.
- atomic : one outer parentheses.

& (i) ✓ (ii) doubling up on parentheses preserves property of have a single parentheses on outside.

Ex: $\omega = 0 \to 1 \to 2 \to 3 \to \cdots$

by prop, $\mathcal{C}_\omega$ is a simplicial computad, with

- $\mathrm{ob}\, \mathcal{C}_\omega = \mathrm{ob}\, \omega = \{n \geq 0\}$
- 0-arrow from j to k, $j \leq k$ (only interesting case) is a string of composable arrows, determined by a subset T:
$$\{j,k\} \subset T \subset [j,k] = \{t \in \omega \mid j \leq t \leq k\}$$
- 1-arrow " " " in brackets, given by
$$\{j,k\} \subset \underset{\substack{\uparrow \\ \text{bracketing of} \\ \text{parentheses.}}}{T^1} \subset \underset{\nwarrow \text{ string of arrows}}{T^0} \subset [j,k]$$
- n-arrow is an n-bracketed sequence of composable arrows
$$\{j,k\} \subset T^n \subset \cdots \subset \underset{\substack{\uparrow \\ \text{nested seq. (b/c parentheses need to be "well-formed")}}}{T^2} \subset T^1 \subset \underset{\nwarrow \text{ underlying sequence}}{T^0} \subset [j,k]$$

In total, $\mathfrak{C}_\omega(j,k)$ has 2^{k-j-1} vertices.

E.g., $\mathfrak{C}_\omega(0,3)$

$$\begin{tikzcd}
\{0,3\} \arrow[r] \arrow[d] \arrow[dr] & \{0,1,3\} \arrow[d] \\
\{0,2,3\} \arrow[r] & \{0,1,2,3\}
\end{tikzcd}$$

Upshot $\mathfrak{C}_\omega(j,k) = \begin{cases} \emptyset & k<j \\ * & j=k \\ (\Delta^1)^{k-j-1} & k>j. \quad (*) \end{cases}$

Next, an <u>atomic</u> 0-arrow is the case $T^0 = \{j,k\}$

" 1-arrow $T^1 = \{j,k\}$

n-arrow $T^n = \{j,k\}$ (location of outer parentheses.

<u>e.g., atomic n-arrows contain the initial vertex in the cube. (*)</u>

§. Homotopy coherent realization and nerve functors

Inside ω, have $[n] = 0 \to 1 \to \cdots \to n$ full subcategory.

Then, $\mathfrak{C}[n]$ is the homotopy coherent n-simplex.

We have a functor $\Delta \xrightarrow{\mathfrak{C}} s\mathcal{C}at$
$$[n] \longmapsto \mathfrak{C}[n].$$

Def'n:
$$\begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "\mathfrak{C}"] \arrow[r, phantom, "(\perp)"] & s\mathcal{C}at_{\partial\mathcal{S}} \arrow[l, bend left, "\mathbb{N}"] \end{tikzcd}$$
$\mathfrak{C}$ is left adjoint to $\mathbb{N}$.

Homotopy coherent nerve $(\mathbb{N}\mathcal{S})_n := \{\mathfrak{C}[n] \longrightarrow \mathcal{S}\}$ "homotopy coherent diagrams of shape $\mathcal{S}$"

a sCat

"terminal ~~object~~ category"

(recall $\mathrm{Coh}(A, \mathcal{S})_n = \{\mathfrak{C}(A \times [n]) \to \mathcal{S}\}$) $\quad \mathbb{N}\mathcal{S} = \mathrm{Coh}(\mathbb{1}, \mathcal{S})$. "coherent diagrams in the shape of a object".

Homotopy coherent realization (not a std. name in the literature)

$\mathfrak{C}$ is left Kan extension, Yoneda embedding $[n] \longmapsto \Delta^n$.
$$\begin{tikzcd} \Delta \arrow[r, hook, "y"] \arrow[dr, bend right, "\mathfrak{C}"'] & s\mathcal{S}et \arrow[d, dashed] \\ & s\mathcal{C}at \end{tikzcd}$$
meaning: $\mathfrak{C}\Delta^n := \mathfrak{C}[n]$

$\mathfrak{C}X := \operatorname{colim} \mathfrak{C}[m]$'s (being inexplicit here).

Similarly $\Delta \xhookrightarrow{\text{full embedding}} \mathcal{C}at$ yields an adjunction
$$[n] \longmapsto [n] = 0 \to 1 \to \cdots \to n.$$
$$\begin{tikzcd} s\mathcal{S}et \arrow[r, bend left, "\mathrm{Ho}"] \arrow[r, phantom, "\perp"] & \mathcal{C}at \arrow[l, bend left, "\text{``nerve''}"] \end{tikzcd}$$
nerve of A has $\{[n] \to A\}$ as its n-simplices.
$$a_0 \to a_1 \to \cdots \to a_n.$$

Rmk: Last time, defined $\mathcal{B}G \subset \mathcal{C}at$, the usual sSet BG is nerve of $\mathcal{B}G$ as we defined it.

Thm: $\mathcal{C}A = \mathfrak{C}A$. That is, for any category A, the free resolution is isomorphic to homotopy coherent realization of the nerve.

Rmk: doesn't just follow from case of $[n]$ by "taking colims" as these functors do, & preserve some colims.

Use:

<u>Prop</u>: (Dugger–Spivak): for any $X \in sSet$, $\mathfrak{C}X$ is also a simplicial computad, with

- $\mathrm{ob}\,\mathfrak{C}X = X_0$
- atomic 0-arrows are 1-simplices $x \xrightarrow{f} y$
- atomic 1-arrows are non-degenerate k-simplices

$$x \overset{\text{first}}{\ } \to 1 \to \cdots \to y^{\text{last}}$$

(**) Moreover $X \hookrightarrow Y$ sSet inclusion $\Downarrow$ $\mathfrak{C}X \hookrightarrow \mathfrak{C}Y$ inclusion of simplicial computads.

- atomic n-arrows are k-simplices $x \to y$, $k > n$, along with

(✡) $\{0,k\} = T^n \subsetneq T^{n-1} \subsetneq \cdots \subsetneq T^1 \subsetneq T^0 = [0,k]$, i.e., an atomic n-arrow in $\mathfrak{C}\Delta^k(0,k)$ that's not in any face.

<u>proof of main thm</u> [sketch]: Both $\mathcal{C}A$ and $\mathfrak{C}A$ have $\mathrm{ob}\,A$ as objects (in case of free resolution, by def'n, in other case of coherent realization by above).

- atomic 0-arrows are morphisms in A = 1-simplices in nerve.
- atomic 1-arrows are k-composable morphisms (once after bracketed) in A = a k-simplex in nerve.
- atomic n-arrows " " " + n-bracketings = a k-simplex plus (✡).

(plus check face + degeneracies).

<u>Cor</u>: A homotopy coherent diagram of shape A in $\mathcal{S}$ is

(i) $\mathfrak{C}A \to \mathcal{S}$ simplicial functor. OR $A \longrightarrow N\mathcal{S}$ simplicial map (implicitly "nerve of A"). (b/c $\mathfrak{C} \dashv N$ adjunction).

(ii) $\mathrm{Coh}(A, \mathcal{S}) \cong (N\mathcal{S})^A$ (e.g., take <u>internal hom</u> in sSet from A to $N\mathcal{S}$)

<u>Thm</u>: (Cordier–Porter): If $\mathcal{S}$ is a Kan complex enriched simplicial category, then $N\mathcal{S}$ is a quasi-category.

<u>Proof</u>:

$$\begin{tikzcd} \Lambda^n_k \arrow[r, hook] \arrow[d, hook] & N\mathcal{S} \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \quad \overset{\text{adjunction}}{\longleftrightarrow} \quad \begin{tikzcd} \mathfrak{C}(\Lambda^n_k) \arrow[r] \arrow[d, hook] & \mathcal{S} \\ \mathfrak{C}(\Delta^n) \arrow[ur, dashed] & \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} \mathfrak{C}\Lambda^n_k(0,n) \arrow[r] \arrow[d, hook] & \mathrm{Map}(x, \\ \mathfrak{C}\Delta^n(0,n) \arrow[ur, dashed] & \end{tikzcd}$$

$0 < k < n$ (<u>inner horn</u>)

"cubical horn"

main thing: to check, by (**), is!

Kan cplx

Now, cover by simplices, & apply property of Kan glo "cube"

Now, cover by simplices & apply Kan cplx properties:

Thm: (Boardman - Vogt):

Coh$(A, \mathcal{S})$ is a quasi-cat if $\mathcal{S}$ is Kan cplx enriched.

Proof: Coh$(A, \mathcal{S}) \cong N\mathcal{S}^A$ + quasi-cats. are exponential ideal

(means property of being a quasi-cat is inherited by mapping in).

To explain slogan: given $\mathcal{S}$ cat-enriched in Kan cplexes, & X, then simplicial maps

$$\begin{tikzcd} X \arrow[r] \arrow[d, leftrightarrow, "\wr"] & N\mathcal{S} \\ \mathfrak{C}X \arrow[r] & \mathcal{S} \end{tikzcd}$$

sSet

htopy coherent diagram of shape X.

& remark: every q-cat is equivalent to a q-cat of the form $N\mathcal{S}$.

$\Rightarrow$ every map of simplicial sets is automatically "homotopy coherent."

$$sSet \underset{Sing}{\overset{|-|}{\rightleftarrows}} Top \qquad Sing(X) \text{ is a Kan complex.}$$

$\Delta^1 = 0 \to 1$ not a Kan complex

Instead, "S^∞" or "$\mathbb{I}$" $= 0 \rightleftarrows 1$ & $0 \nearrow 1 \searrow 0$ with $0 = 0$ ($\simeq$), $1 \nearrow 0 \searrow 1$ with $1 = 1$ ($\simeq$)

& higher.