

E. Rehl II

Aim: diagrams in a \mathcal{C} cat. are automatically homotopy coherent. — reconceptualize \mathcal{C} .

Last time: a homotopy coherent diagram of shape A is

$$\mathcal{C} \rightarrow \text{Space}, \text{ i.e. } a \in A \mapsto X_a \in \text{Space}$$

$$\mathcal{C}(A(a,b)) \rightarrow \text{Map}(X_a, X_b)$$

Set (in a Kan cplx, paths are "invertible"),

Today: Let \mathcal{S} be any category enriched in spaces Kan complexes. "locally Kan categories?"

S. Simplicial Computads

$f: x \rightarrow y$ is atomic if it can't be factored (& isn't id_x).

A category is freely generated (by a (reflexive directed) graph of atomic arrows) iff every arrow admits a unique factor uniquely into atomics.

Def (Simplicial computad): A simplicial category $A_0 = (A_0 \rightrightarrows A_1 \rightrightarrows A_2 \dots \text{ all } v/ob A)$

is a simplicial computad if

(i) each A_n is freely generated (by "atomic n -arrows" associated)

(ii) for each $[n] \rightarrow [m]$ in Δ the functor $A_n \rightarrow A_m$ preserves reflexive directed graphs.

Prop: $\mathcal{C}A$ is a simplicial computad.

Proof: $\mathcal{C}A_0 =$

$$Fu A \begin{matrix} \xrightarrow{Fu_1} \\ \xleftarrow{Fu_2} \end{matrix} (Fu)^2 A \begin{matrix} \xrightarrow{Fu_1} \\ \xleftarrow{Fu_2} \end{matrix} (Fu)^3 A \dots$$

- 0 arrows are strings of composable morphisms

- atomic 0-arrows are single arrows.

- 1-arrows are " " in 1 set of parentheses.

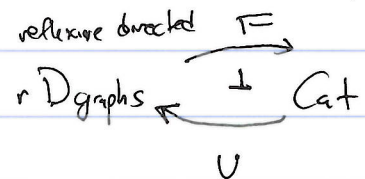
$$(fgh)(k)(e)(m n)$$

- Atomic 1-arrows have form (-string).

- n arrows ——— n -parenthesized.

- atomic : one outer parentheses.

& (i) \leftarrow (ii) doubling up on parentheses preserves property of have a single parentheses on outside.



Ex: $\omega = 0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow \dots$

by prop, $\mathcal{C}\omega$ is a simplicial complex, with

- $ob \mathcal{C}\omega = ob \omega = \{n \geq 0\}$

- 0-arrow from j to k , $j \leq k$ (only interesting case) is a string of composable arrows, determined by a subset T :

$$\{j, k\} \subset T \subset [j, k] = \{t \in \omega \mid j \leq t \leq k\}$$

- 1-arrow " " " in brackets, given by

$$\{j, k\} \subset T^1 \subset T^0 \subset [j, k]$$

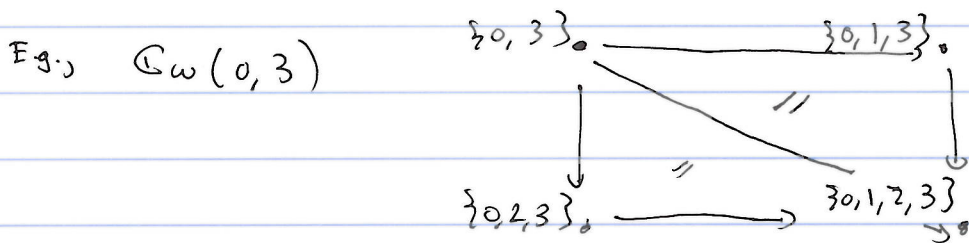
\uparrow location of parentheses. \uparrow string of arrows

- n -arrow is an n -bracketed sequence of composable arrows

$$\{j, k\} \subset T^n \subset T^{n-1} \subset T^1 \subset T^0 \subset [j, k]$$

\uparrow nested seq. (k-lc parentheses need to be "well-formed")

In total, $\mathcal{C}\omega(j, k)$ has 2^{k-j-1} vertices.



$$\text{Upshot } \mathcal{C}\omega(j, k) = \begin{cases} \emptyset & k < j \\ * & j = k \\ (\Delta)^{k-j-1} & k \geq j \end{cases} \quad (*)$$

Next, an atomic 0-arrow is the case $T^0 = \{j, k\}$

" 1-arrow $T^1 = \{j, k\}$

" n -arrow $T^n = \{j, k\}$ (location of ~~at~~ outer parentheses)

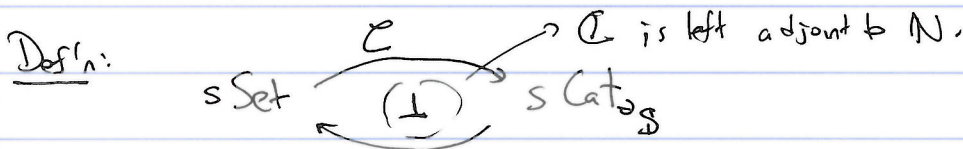
e.g., atomic n -arrows contain the initial vertex in the cube. (*)

S. Homotopy coherent realization and nerve functors

Inside ω , have $[n] = 0 \rightarrow 1 \rightarrow \dots \rightarrow n$ full subcategory.

Then, $\mathcal{C}[n]$ is the homotopy coherent n-simplex.

We have a functor $\Delta \xrightarrow{\mathcal{C}} \mathcal{S}Cat$
 $[n] \hookrightarrow \mathcal{C}[n]$.

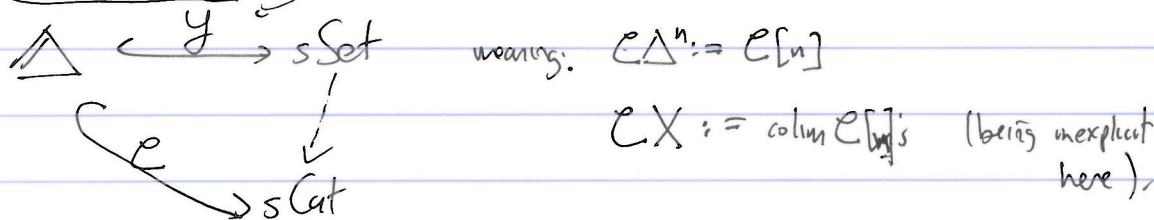


Homotopy coherent nerve $(N\mathcal{S})_n := \{ \mathcal{C}[n] \rightarrow \mathcal{S} \}$ "homotopy coherent diagrams of shape \mathcal{S} "
 "terminal object category".

(recall $\text{Coh}(A, \mathcal{S})_n = \{ \mathcal{C}(A; [n]) \rightarrow \mathcal{S} \}$) $N\mathcal{S} = \text{Coh}(\underline{1}, \mathcal{S})$. "coherent diagrams in the shape of a object".

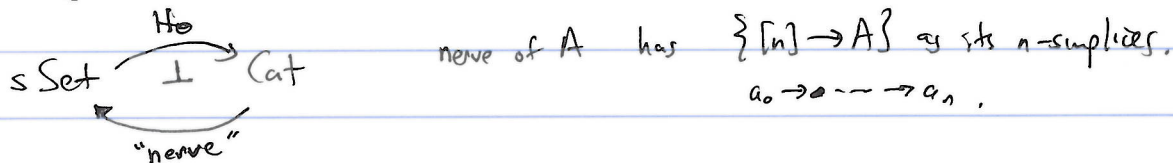
Homotopy coherent realization (not a std. name in the literature)

\mathcal{C} is left Kan extension of Yoneda embedding $[n] \hookrightarrow \Delta_n$.



Similarly $\Delta \xleftrightarrow{\text{filling embedding}} \text{Cat}$ yields an adjunction

$[n] \hookrightarrow [n] = 0 \rightarrow 1 \rightarrow \dots \rightarrow n$



Remark: Last time, defined $BG \subset \text{Cat}$, the usual $\mathcal{S}Set$ BG is nerve of BG as we defined it.

Then: $\mathcal{C}A = \mathcal{C}A$. That is, for any category A , the free resolution is isomorphic to homotopy coherent realization of the nerve.

Remark: doesn't just follow from case of $[n]$ by "taking colims" as term functor do, if prove some colimits.

Now, over the simplices & apply Kan cplx properties:

Thm: (Boardman-Vogt):

$\text{Coh}(A, \mathcal{S})$ is a quasi-cat. if \mathcal{S} is Kan cplx enriched.

Proof: $\text{Coh}(A, \mathcal{S}) \cong \mathbb{N}\mathcal{S}^A$ + quasi-cats. are exponential ideal
(means property of being a quasi-cat. is inherited by mapping in).

To explain slogan: \mathcal{S} cat-enriched is Kan cplx, & X , then simplicial maps.

$$\begin{array}{ccc} \text{sSet} \rightarrow & X & \longrightarrow \mathbb{N}\mathcal{S} \\ & \downarrow & \\ & CX & \longrightarrow \mathcal{S} \end{array}$$

htopy coherent diagram of shape X .

Remark: every qcat is equivalent to a qcat of the form $\mathbb{N}\mathcal{S}$.

\Rightarrow every map of simplicial sets is automatically "homotopy coherent."

$$\text{sSet} \xrightleftharpoons[\text{Sing}]{|-|} \text{Top} \quad \text{Sing}(X) \text{ is a Kan cplx.}$$

$\Delta^1 = 0 \rightarrow 1$ not a Kan cplx
Farkas, " S^∞ " or " $\mathbb{R}^k = 0 \rightleftharpoons 1$ "

$$\delta \cdot 0 \xrightarrow{\sim} 1, \quad \delta \cdot 0 \xrightarrow{\sim} 1$$

δ higher.