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Last time: nerve of a cat ⇝ shape of a htpy coherent diagram in a quasi-category.

"local nerve" of a 2-cat* ⇝

shape of htpy coherent categorical structure in a quasi-cat. enriched category.

*: but, want simplicial cat on morphism spaces to be a simplicial computad.

Story so far

cat A; a htpy coherent diagram in S of shape A was: $\mathfrak{C}A \to S$ or: $A \xrightarrow{\text{simplicial map}} NS$ ← quasi-cat.
↑ nerve of A

Kan complex enriched category S has objects X, Y, & mapping spaces $\mathrm{Map}(X,Y)$ ← Kan cplx. (0-arrows ⇝ functions $X \to Y$, n-arrows ⇝ "n-htpies")

Quasi-categorically-enriched $\mathbb{K}$ — 0 arrows ⇝ functors
- objects A, B, etc, function complexes $\mathrm{Fun}(A,B)$ ← quasi-category — 1-arrows ⇝ nat. trans.
n-arrows ⇝ n-htpies.

Ex: $\mathbb{K} = \mathbb{Q}Cat$, CSS ← complete Segal spaces, $Segal$, $\mathcal{C}at$, fibered versions,

q. cats are enriched over themselves

In this lecture: ∞-category ≡ object in some $\mathbb{K}$.

& "quasi-category" ≡ simplicial set w/ inner horn fillers (ex. of an ∞-cat).

§ Monads

Def: A monad on a category B is $B \xrightarrow{T} B$ together with $\mathrm{id}_B \xrightarrow{\eta} T$, $T^2 \xrightarrow{\mu} T$

s.t.

$$\begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, equal] \\ & T & \end{tikzcd} \qquad \begin{tikzcd} T^3 \arrow[r, "\mu T"] \arrow[d, "T\mu"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \quad \text{in } \mathrm{Fun}(B,B)$$

Ex: $P = \{P_n\}_{n \in \mathbb{N}}$ operad ⇝ monad $T(b) := \sum_{n \in \mathbb{N}} P_n \times_{\Sigma_n} b^n$ (provided these symbols (limits, colimits) make sense)

P-algebras ≡ T-algebras.

A monad is a 2-categorical diagram in $\mathbb{C}\mathrm{at}$ $(\mathbb{M}\mathrm{nd} \longrightarrow \mathbb{C}\mathrm{at})$.

Q: What's a homotopy-coherent monad?

$$\mathbb{M}\mathrm{nd} \longrightarrow \mathbb{K}, \quad \mathbb{M}\mathrm{nd} \text{ is a simplicial cat w/}$$

- object "$+$"
- atomic 0-arrow "$t : + \to +$", 0 arrows $t^n : + \to +$ $n \geq 0$, $t^0 = id_+$.
- atomic 1-arrows $(\eta : \cancel{id_+ \to T}, \ \mu : \cancel{t^2 \to t})$ biased view instead:

$$\mu_n : t^n \to t \quad n \geq 0$$

where $\mu_0 = \eta$, $\mu_1 = id_+$, $\mu^2 = \mu$, ...

can compose: $\mathrm{Mnd}(+,+) \times \mathrm{Mnd}(+,+) \to \mathrm{Mnd}(+,+)$

- composite 1-arrows $\mu_{i_1}, \ldots, \mu_{i_m}$ ⇝ composite gives $t^{\sum_j i_j} \longrightarrow t^m$.

$$\mu[\alpha] := \begin{tikzcd} + \arrow[r, bend left, "t^{i_1}"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow \mu_{i_1}"] & + \arrow[r, bend left, "t^{i_2}"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow \mu_{i_2}"] & + \arrow[r, bend left] \arrow[r, bend right, "t"'] & + \cdots + \arrow[r, bend left, "t^{i_m}"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow"] & + \end{tikzcd}$$

where $[n-1] \xrightarrow{\alpha} [m-1] = \{0, \ldots, m-1\}$

↑ surjection whose fiber of j, $\alpha^{-1}(j) = i_j$.

- atomic 2-arrows witness the relations:

this commutes:

$$\begin{tikzcd} & t^m \arrow[dr, "\mu^m"] & \\ t^n \arrow[ur, "\mu[\alpha]"] \arrow[rr, "\mu^n"'] & & t \end{tikzcd}$$

- 2-arrows are:

$$\begin{tikzcd} & t^m \arrow[dr, "\mu[\beta]"] & \\ t^n \arrow[ur, "\mu[\alpha]"] \arrow[rr, "\mu[\gamma]"'] & & t^k \end{tikzcd} \quad \text{whenever} \quad \begin{tikzcd} & {[m-1]} \arrow[dr, "\beta"] & \\ {[n-1]} \arrow[ur, "\alpha"] \arrow[rr, "\gamma"'] & & {[k-1]} \end{tikzcd}$$

etc.

Summary: $\underline{\mathrm{Mnd}(+,+)}$ = (nerve of) Δ_+ : finite ordinals $[-1]$, $[0]$, $[1]$, ... + maps.

empty ordinal (hence "+")

Def: (free htpy coherent monad)

$\mathbb{M}\mathrm{nd}$ is a simp. cat with one object $+$ with $\mathbb{M}\mathrm{nd}(+,+) = \Delta_+$ and

$$\begin{array}{ccc} \mathbb{M}\mathrm{nd}(+,+) \times \mathbb{M}\mathrm{nd}(+,+) & \longrightarrow & \mathbb{M}\mathrm{nd}(+,+) \\ \| & & \\ \Delta_+ \times \Delta_+ & \xrightarrow{\oplus} & \Delta_+ . \end{array}$$

Prop: (Lawvere) 2-functors $\mathbb{M}\mathrm{nd} \to \mathcal{C}at$ are monads.

Def: A homotopy coherent monad is a simplicial functor

$$\mathbb{M}\mathrm{nd} \longrightarrow \mathbb{K}, \text{ i.e.,}$$

$$+ \longmapsto B$$

$$\Delta_+ = \mathbb{M}\mathrm{nd}(+,+) \longrightarrow \mathrm{Fun}(B,B) \quad \text{htpy coherent diagram}$$

$$\begin{tikzcd} id_B \arrow[r, "\eta"] & T \arrow[r, shift left=2, "\eta T"] \arrow[r, shift right=2, "T\eta"'] & T^2 \arrow[l, "\mu" description] \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] & T^3 \arrow[l, shift left=1.5] \arrow[l, shift right=1.5] \cdots \end{tikzcd} \in \mathrm{Fun}(B,B)$$

htpoh monad resolution.

§ Adjunctions

Def'n: An adjunction is A, B, functors $A \xrightarrow{U} B$, $B \xrightarrow{F} A$,

$id_B \xrightarrow{\eta} UF \quad FU \xrightarrow{\varepsilon} id_A$ s.t.

$$\begin{tikzcd} & UFU \arrow[dr, "U\varepsilon"] & \\ U \arrow[ur, "\eta U"] \arrow[rr, equal, "id"'] & & U \end{tikzcd} \in \mathrm{Fun}(A,B) \quad \&$$

$$\begin{tikzcd} & FUF \arrow[dr, "\varepsilon F"] & \\ F \arrow[ur, "F\eta"] \arrow[rr, equal, "id"'] & & F \end{tikzcd} \quad \mathrm{Fun}(B,A)$$

Lem: Adj. $\rightsquigarrow$ monad on B, namely $(UF, \eta, U\varepsilon F)$.

The free adjunction $\mathcal{A}dj$:

$$\begin{tikzcd} + \arrow[loop left, "\Delta_+"] \arrow[r, bend left, "\Delta_\infty"] & - \arrow[l, bend left, "\Delta_\infty"] \arrow[loop right, "\Delta_+^{op}"] \end{tikzcd}$$

"universal comonad"

$\mathcal{A}dj(+,+) = \Delta_+$

$\nwarrow$ c.f. Lurie's notation

Prop: (Schanuel-Street): 2-functors $\mathrm{Adj} \to \mathcal{C}at$ are adjunctions.

Prop: (R.-Verity): Adj is a simplicial computad, with

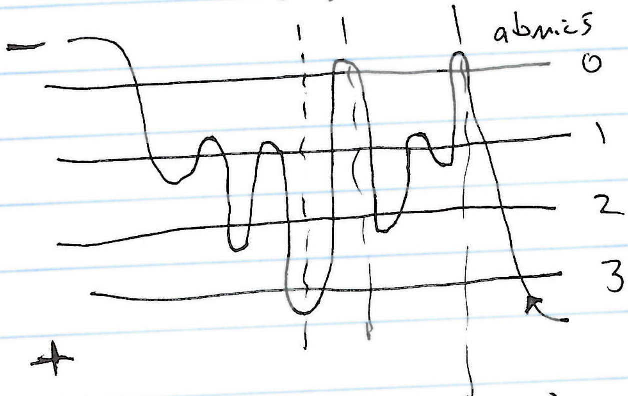
- objects $+, -$
- atomic 0-arrows $f: + \to -$ $\quad u: - \to +$.

(spells out the generating data of a htpy-coherent adjunction).

- n-arrows are: strictly undulating squiggles on $(n+1)$ lines.

atomics (don't hit +/- except at end

e.g., 3-arrow:



$\in \mathrm{Adj}_3(+,-)$

3-arrow

(face maps: remove lines & "straighten out").

Define: A htpy coherent adjunction is a simplicial functor $\mathrm{Adj} \to \mathcal{K}$, i.e.,

$- \longmapsto A$

$+ \longmapsto B$,

4-htpy coherent diagrams

$\underbrace{\Delta_+ \to \mathrm{Fun}(B,B), \quad \Delta_+^{op} \to \mathrm{Fun}(A,A)}_{(*)}, \quad \underbrace{\Delta_\infty \to \mathrm{Fun}(A,B)}_{(**)}, \quad \text{---}$

(*)
$$\mathrm{id} \xleftarrow{\varepsilon} FU \underset{FU\varepsilon}{\overset{\varepsilon FU}{\leftleftarrows}} \hspace{-1em} \xrightarrow{F\eta U} FUFU \;\overset{\leftarrow}{\underset{\leftarrow}{\leftleftarrows}}\; FUFUFU$$

(**)
$$U \underset{U\varepsilon}{\overset{\eta U}{\rightleftarrows}} UFU \underset{}{\overset{\eta UFU}{\rightleftarrows}} UFUFU \;\text{-}\sim$$

$$\begin{tikzcd} s\mathrm{Set} \arrow[r, bend left, "\mathrm{Ho}"] \arrow[r, phantom, "\perp"] & \mathrm{Cat} \arrow[l, bend left, "\text{nerve}"] \end{tikzcd}$$

Today:

$$\begin{tikzcd} s\mathrm{Cat} \equiv s\mathrm{Set}\text{-}\mathrm{Cat} \arrow[r, bend left, "\text{local Ho}"] \arrow[r, phantom, "\perp"] & \mathrm{Cat}\text{-}\mathrm{Cat} \equiv 2\text{-}\mathrm{Cat} \arrow[l, bend left, "\text{local nerve}"] \end{tikzcd}$$

$\mathbb{K}$ quasi-cat. enriched $\rightsquigarrow$ homotopy 2-cat $\mathrm{Ho}\,\mathbb{K}$ (by taking local Ho everywhere)

<u>Thm:</u> (R-verity) $\mathbb{K}$ quasi-cat.-enriched category. Then any adjunction in $\mathrm{Ho}\,\mathbb{K}$ extends to a htpy coherent adjunction on $\mathbb{K}$.

e.g., functors, unit, counit, ~~&~~ & ~~tr~~ diagrams holding up to htpy

("uniquely, in a htopical manner") but have to make an "odd # of fixed choices" e.g., F, U, η, or $F, U, \eta, \varepsilon, \delta$ are fixed, not F, U, η, ε

(Rmk: this is not true for monads; but if it came from an adjunction, then can lift to a htpy coherent monad).

(uses a "universal property" of the counit; sth. which doesn't hold for monads).