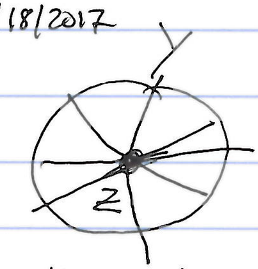
P. Seidel, Rings of definition of Fukaya categories of CY hypersurfaces 7/18/2017



X symplectic manifold, $c_1(X) = [\omega_X]$ (Fano, or monotone)

$Y \subset X$ anticanonical hypersurface, $c_1(Y) = 0$.

Assume: Y is the member of a Lefschetz pencil of such hypersurfaces, with base locus Z.

$\mathcal{F}(Y)$ Fukaya category, $\mathbb{Z}$-graded A_∞ category over $\mathbb{C}((q))$.

We also have: the relative Fukaya category

$$\underset{\text{ample divisor}}{\mathcal{F}(Y, Z)} \otimes_{\mathbb{C}[[q]]} \mathbb{C}((q)) \hookrightarrow \mathcal{F}(Y)$$
, and the "affine" Fukaya category,

$\mathcal{F}(Y \setminus Z)$, over $\mathbb{C}$, with

$$\mathcal{F}(Y, Z) \otimes_{\mathbb{C}[[q]]} \mathbb{C} \hookrightarrow \mathcal{F}(Y \setminus Z).$$

Vanishing cycles for the pencil form full subcategories

$$\mathcal{B} \subset \mathcal{F}(Y \setminus Z), \quad \mathcal{B}_q \subseteq \mathcal{F}(Y, Z).$$

Question: Over which subring of $\mathbb{C}[[q]]$ is $\mathcal{B}_q$ defined? (meaning, can make choices so defined over this ring)

Application: If $\mathcal{B}_q$ is defined over $R \subset \mathbb{C}[[q]]$, the full subcategory of $\mathcal{F}(Y)$ consisting of $L \subset Y$ with $H^1(L) = 0$ is defined over the algebraic closure $\overline{R} \subset \mathbb{C}((q))$.

This a general algebro-geometric principle.

Ex: Let M be a cplx. proj. variety that is defined over $\overline{\mathbb{Q}}$. If $E \to M$ is a vector bundle with $\mathrm{Ext}^1(E, E) = 0$, then E is defined over $\overline{\mathbb{Q}}$. (or rather, is isomorphic to one which is).

Being "defined over R" is not necessarily ~~well-behaved up~~ under homotopically well-behaved.

Ex: Take chain complexes

$$h \circlearrowright (C, d) \underset{g}{\overset{f}{\leftrightarrows}} (\tilde{C}, \tilde{d}) \quad \text{w/ } g \circ f = id \ \& \ f \circ g \simeq id$$

$$\quad f\circ g = \tilde{d}h + h\tilde{d} + id.$$

Take a deformation $d_q = d + O(q)$ of d, and transfer it to $\tilde{C}$. Explicit formula:

$$\tilde{d}_q = \tilde{d} + g\,(d_q - d)\,f + g\,(d_q - d)\,h\,(d_q - d)\,f + \cdots$$ (infinite sum!)

→ If d_q is defined over $R \subset \mathbb{C}[[q]]$, the same is not necessarily true for $\tilde{d}_q$.

(maybe ok if cplxes are finite-dimensional, b/c terms are eventually 0)

Ex: Let $\mathcal{B}, \tilde{\mathcal{B}}$ be ~~isomorphic~~ minimal A_∞ algebras over $\mathbb{C}$ that are A_∞-isomorphic. If $\mathcal{B}_q$ is a minimal deformation of $\mathcal{B}$ defined over R, one can find a corresponding deformation $\tilde{\mathcal{B}}_q$ with the same property.

(in this case: can show finiteness of sums appearing in transfer situation)

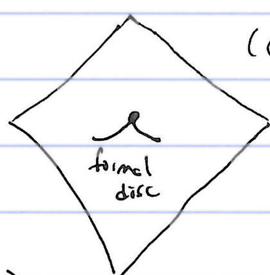
(unfortunately, this doesn't apply in our case here. So, have to be careful, and/or think of more structured models.)

Conjecture: Let $r = \dim H^2(X)$ (the Fano). Then there are explicitly given $g_1, \dots, g_{r-1} \in \mathbb{C}[[q]]^\times$ (invertible), & $f \in \mathbb{C}[[q]]$, $f(0) = 0$ (not invertible), $f'(0) \neq 0$ (so $= aq + \cdots$)

such that $\mathcal{B}_q$ is defined over $\mathbb{C}[g_1^{\pm 1}, \dots, g_{r-1}^{\pm}, f] \subseteq \mathbb{C}[[q]]$.

Geometric picture:

$(\mathbb{C}^\times)^{r-1} \times \mathbb{C}$



Recall that (a basis of) vanishing cycles are ordered, $(V_1, \dots, V_m)$, &

$$\mathcal{B} = \bigoplus_{i,j=1}^{m} CF^*(V_i, V_j)$$

$\deg(e_i) = 0$, $\deg(p_i) = n$

V_i is a sphere, so I can assume $CF^*(V_i, V_i) = \mathbb{C} \cdot e_i \oplus \mathbb{C} \cdot p_i$.

Then, $\mathcal{B} = \mathcal{A} \oplus \mathcal{P}$.

where:

$$\mathcal{A} = \bigoplus_{i<j} CF^*(V_i, V_j) \oplus \bigoplus_i \mathbb{C}\cdot e_i \quad \text{an } A_\infty \text{ subalgebra}$$

and

$$\mathcal{P} = \bigoplus_{i>j} CF^*(V_i, V_j) \oplus \bigoplus \mathbb{C}\cdot p_i \quad \text{and similarly}$$

$\mathcal{B}_q = \mathcal{A}_q \oplus \mathcal{P}_q$. Let's say $\mathcal{A}$ has "weight 0" and $\mathcal{P}$ has "weight -1".
Then the A_∞ structure has pieces of weights $0, 1, 2, 3, \ldots$

(weight 0: $\mathcal{A}^{\otimes r} \to \mathcal{A}$ & $\mathcal{A}^{\otimes r} \otimes \mathcal{P} \otimes \mathcal{A}^{\otimes s} \to \mathcal{P}$ weight 1: $\mathcal{A}^{\otimes r} \otimes \mathcal{P} \otimes \mathcal{A}^{\otimes s} \to \mathcal{A}$, etc.)

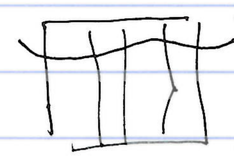
Then,

Conjecture: the weight w part of $\mathcal{B}_q$ is a polynomial in f of degree $\leq w$ whose coefficients lie in $\mathbb{C}[g_1^{\pm 1}, \ldots, g_{r-1}^{\pm 1}]$.

$\boxed{r=1} \Rightarrow \mathcal{A}_q$ is the trivial deformation of $\mathcal{A}$ (Auroux-Katzarkov-Orlov), the only information lies in f.

Let $\tilde{X}$ be the blowup of X along Z, the base locus.

$\mathbb{P}^1 \times Z = \hat{Z}$



$\hat{X}$

$$Y \longrightarrow \hat{X} \longrightarrow \mathbb{P}^1$$

$H^2(\hat{X}) = \mathbb{C}[Y] \oplus \mathbb{C}[\hat{Z}]$.

Consider GW invt. counting sections (spheres in $\hat{X}$ w/ degree 1 over $\mathbb{P}^1$), and set "almost $[\omega]$": actual $[\omega]$ is $[\hat{Z}]$ + mult. of fiber.

$$z^{(1)} = \sum_{\substack{A \in H_2(\hat{X}) \\ A\cdot[Y]=1}} z_A\, q^{A\cdot[\hat{Z}]} \in H^2(\hat{X}; \mathbb{C}((q)))$$

only neg. term: constant sections have $A\cdot[\hat{Z}] = -1$.

(b/c $[\hat{Z}]\cdot[\hat{Z}] =$

Similarly, consider bisections $\boxed{z^{(2)} \in H^0(\hat{X}; \mathbb{C}[[q]]) = \mathbb{C}[[q]]}$

Write

$$q^{-1}[\hat{Z}] = \psi z^{(2)} - \eta[Y], \text{ where } \boxed{\psi \in \mathbb{C}[[q]],\ \psi = 1 + O(q),\ \eta \in \mathbb{C}[[q]]}$$

(no problem b/c $z^{(1)}$ has const. solutions, so $\psi_0 \neq 0$, e.g., ψ invertible)

Conjecture, continued: for $r=1$, f is a solution of:

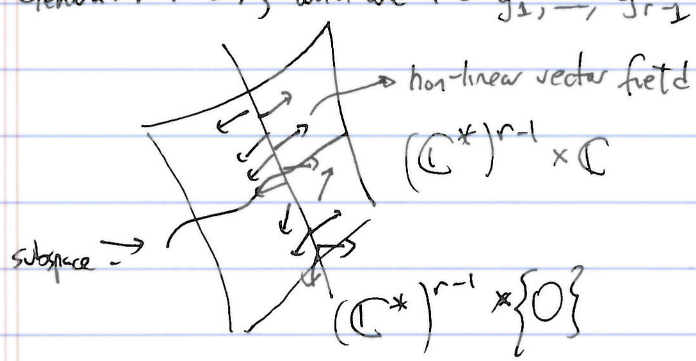
$$Sf + 8z^{(2)}\psi^2 + \frac{1}{2}\left(\eta - \frac{\psi'}{\psi}\right)^2 + \left(\eta - \frac{\psi'}{\psi}\right)' = 0 \quad (*)$$

where $S = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$ is the Schwarzian non-linear differential operator. ($Sf = ?$)

(this characterizes f nearly uniquely). (secretly f is a quotient of solutions of 2nd order ODEs)

(complete intersections: expect "as many f's" as codimension)

General r (≥ 1), what are the $g_1, \dots, g_{r-1}$? (Again f is obtained by solving a generalization of a diff. eq. like $(*)$).



We make a formal coordinate transformation to "linearize" the vector field.

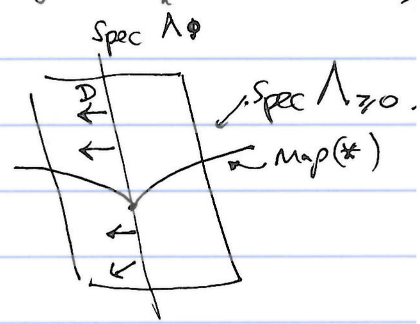
Take $H = H_2(\hat{X})/\text{torsion} \cong \mathbb{Z}^{r+1}$ with grading $H^i = \{A : A \cdot [Y] = \frac{i}{2}\}$ (replaces c_1 exactly)

and filtration (replacement for $[\omega]$ (approx.))

$$H_{\geq k} = \{A : A \cdot [\hat{Z}] \geq k\}$$ & associated graded $H_k = \{A : A \cdot [\hat{Z}] = k\}$

Associated graded Novikov rings

$$\Lambda \supseteq \Lambda_{\geq 0} \longrightarrow \Lambda_0$$



(graded, so have extra $\mathbb{C}^*$ actions).

We have a map $\Lambda_{\geq 0} \longrightarrow \mathbb{C}[t, t^{-1}][[q]]$ $(*)$

$$q^A \longmapsto t^{A \cdot [Y]} q^{A \cdot [\hat{Z}]}$$

$\Lambda_{\geq 0}$ has a distinguished derivation
$$D q^B = \sum_{A \cdot [Y] = 1} (B \cdot z_A) \, q^{A+B}.$$

GW invt. from before, contributing to $z^{(1)}$.

("straightening" the vec. field)

<u>Lemma</u>: Take $\Lambda_0 [[\tilde{q}]]$, $\deg(\tilde{q}) = -2$. There is a unique graded (filtered) isomorphism

$$\begin{tikzcd} \Lambda_{\geq 0} \arrow[r, "\cong"] \arrow[d, "D \ \deg +2"'] \arrow[dr, phantom, "\circlearrowright"] & \Lambda_0 [[\tilde{q}]] \arrow[d, "\partial_{\tilde{q}}"] \\ \Lambda_{\geq 0} \arrow[r, "\cong"] & \Lambda_0 [[\tilde{q}]] \end{tikzcd}$$

(s.t. at $q = 0$, this is the identity map).

Take $\Lambda_0 [[\tilde{q}]] \xleftarrow{\cong} \Lambda_{\geq 0} \xrightarrow{(*)} \mathbb{C}[t, t^{-1}][[q]]$

$\Rightarrow$ yields $g_1, \ldots, g_{r-1}$ (take fns. on Λ_0, degree 0 part, $r-1$ generators, map to the right).

(note, no $z^{(2)}$ needed for $g_1, \ldots, g_{r-1}$).