H-Tanaka, Morse theory and a stack of broken lines (joint w/ J. Lurie)

Exercise: Fix $Y_0, Y_1, \dots, Y_n \in \mathcal{C}$ (chain complexes or spectra or a stable ∞-cat.)

set $A := \bigoplus_{j>i} \hom(Y_j, Y_i)$

$$\begin{pmatrix} 0 & \hom(Y_1,Y_2) & \hom(Y_0,Y_2) & \dots \\ 0 & 0 & \hom(Y_0,Y_1) & \dots \\ \vdots & \dots & 0 & \dots \end{pmatrix}$$

non-unital assoc. alg. ~~[illegible]~~ . .

and $\mathcal{S} := \begin{pmatrix} 0 & \mathbb{S}[1] & & \\ & \ddots & \mathbb{S}[1] & 0 \\ & & \ddots & \ddots \end{pmatrix}$ note $\mathcal{S}^2 = 0$.

super-Δ:

& check: Space of maps $\{\mathcal{S} \to A\} \simeq \left\{ \begin{matrix} X_0 \to X_1 \to \dots \to X_n \\ \text{with } X_{i+1}/X_i \simeq Y_i \end{matrix} \right\}$

Morse theory: X_i ob of Floer category

so $\mathrm{Map}(\mathcal{S}, A)$ picks out morphisms $\alpha \in \hom(Y_1, Y_2)$, $\beta \in \hom(Y_0, Y_1)$

$\cong$ & $\gamma \in \hom(Y_0, Y_2)$ w/ $\partial\gamma = \beta \circ \alpha$.

$\Gamma_c^{\otimes}(\text{Broken}, \mathcal{F})$

factorizable cptly supp. sections. stack a sheaf: $f_!$ ~~sheaf~~ on moduli of flowlines

moduli $\xrightarrow{f}$ Broken .

Intro: It's important to add ∂ strata. Fix $X \simeq pt. \longrightarrow \mathbb{R}$ Morse function.

$$\mathcal{M}(X) = \{\gamma : \mathbb{R} \to X\}/\mathbb{R} = pt/\mathbb{R} = B\mathbb{R} \quad \text{(as a stack)}$$

don't throw out constant trajectories

mult: Broken $\times$ Broken $\xrightarrow{\text{concat.}}$ Broken (stacked multiplication)

to cptify: $\{\gamma_1, \gamma_2 : \mathbb{R} \to X\}/\mathbb{R}^2 = pt/\mathbb{R}^2 = B\mathbb{R}^2$ (as stack)

so "$\overline{\mathcal{M}(pt)}$" $\simeq \bigcup_{k \geq 1} B\mathbb{R}^k$ (but not a disjoint union, encodes breaking)

$\parallel$
Broken

Motivation:

(1) can actually consider a stack: $\overline{\mathcal{M}}(Y) \longrightarrow$ Broken (via canon. $Y \longrightarrow *$) (Can use to prove Künneth formula in one swoop (Universal for Morse theory).)

(2) Generalizes to Lag. Floer theory. & ~~stable~~ any ∞-cat w/ $\otimes$-structure (factorizable sheaves)

(3) Thm: [Lurie-T.]: $\mathrm{Shv}^{\otimes}(\text{Broken}; \mathcal{C}) \simeq A_\infty \mathrm{Alg}^{nu}(\mathcal{C})$.

(Verdier–Koszul dual to $\cancel{\mathrm{co}}\mathrm{Shv}^{\otimes}(\mathrm{Ran}(\mathbb{R}; \mathcal{C}) \overset{??}{\simeq} \mathrm{AssAlg}^{nu}(\mathcal{C})$??)

__Stacks: by example:__

Fix top. group G. ~~Define~~ a category $\underline{BG}$:

$$\mathrm{ob}\, \underline{BG} = \left\{ \begin{matrix} P \\ \downarrow \\ S \end{matrix} \right\} \leftarrow \text{principal } G\text{-bdle.} \qquad \mathrm{Mor}\, \underline{BG} = \left\{ \begin{matrix} P & \xrightarrow{\tilde f} & P' \\ \downarrow & \cancel{f} & \downarrow \\ S & \longrightarrow & S' \end{matrix} \right\}$$

"category fibred over Top."

$$\underline{BG} \xrightarrow{\pi} \mathrm{Top}$$
$$\begin{matrix} P \\ \downarrow \\ S \end{matrix} \longmapsto S.$$

__Obs:__ (0) $\mathrm{Top}^{op} \longrightarrow \mathrm{Cat}$

$S \longmapsto \pi^{-1}(S, \mathbb{1}_S) = \{G\text{-bdles over } S\}$.

not a functor b/c doesn't satisfy associativity on the nose (only up to coherent natural isomorphisms.)

(1) $\underline{BG}$ looks like a sheaf:

$$\begin{matrix} P \\ \downarrow \\ S \end{matrix} = \begin{matrix} P & \leftarrow & \{P_i\} \\ \downarrow & & \downarrow \\ S & \leftarrow & \{U_i\} \end{matrix}$$

cocycle

__Def of Broken__:

__Def:__ A __broken line__ is a top. space $I \circlearrowleft \mathbb{R}$ action (cont.)

(0) $I \overset{\text{homeo}}{\cong} [0,1]$. (& $\mathbb{R}$ has fixed pts. at boundary points)

(1) $I^{\mathbb{R}}$ = finite set

(2) $\mathbb{R}$ defines a total order on I.

(0) + (1) so far

#(2): get rid of non-directed actions

__Def'n:__ A __family__ of broken lines on S is a top. space $\mathbb{R} \circlearrowright \tilde{S} \xrightarrow{p} S$ s.t.

p. $\mathbb{R}$ equiv. for trivial action on S (fiberwise action)

(0) $\forall x \in S$, $\exists\, U \overset{S}{\subset} \ni x$, & a homeo.

$$p^{-1}(U) \cong U \times [0,1]$$

(1) $\forall x$, $p^{-1}(x)$ is a broken line.

(2) $\tilde{S}^{\mathbb{R}} \xrightarrow{r} S$ is __unramified__

(locally $r^{-1}(U)^{\mathbb{R}} \xrightarrow{r} S$ is a closed embedding)

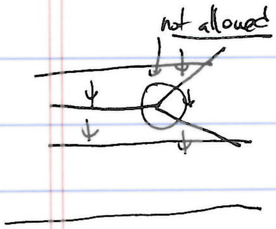
i.e., $\forall x \in S$, $\exists\, U^{\text{open}} \ni x$ s.t.

$$p^{-1}(U)^{\mathbb{R}} \cong k_0 \amalg \cdots \amalg k_n$$

and $k_i \xrightarrow{p} \cancel{S}$ closed embeddings

Non-ex:

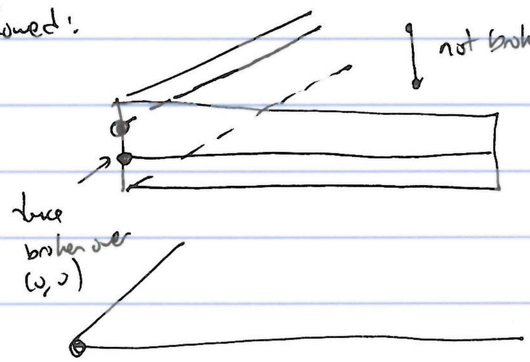
not allowed.



$S = \mathbb{R}$

__Def'n:__ Broken —— (cont'd)

allowed:



$S = \mathbb{R}^2_{\geq 0}$

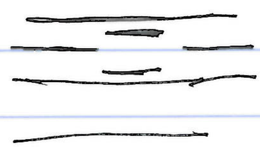
Def'n: Broken is the cat. with

obj $= \left\{ \begin{array}{c} \tilde{S} \\ \downarrow \\ S \end{array} \text{ fam. of broken lines} \right\}$

hom $= \left\{ \begin{array}{ccc} \tilde{S} & \xrightarrow{\tilde{f}} & \tilde{S}' \\ \downarrow & & \downarrow \\ S & \xrightarrow{f} & S' \end{array} \quad \tilde{f} \text{ } \mathbb{R}\text{-equiv homeo on fibers} \right\}$

Prop: Broken $\longrightarrow$ Top makes broken a stack (e.g., it's a sheaf.)

Multiplication: concat:



Lemma: $\exists$ a cover

$$\coprod_{n \geq 0} \mathbb{R}^n_{\geq 0} \longrightarrow \text{Broken}.$$

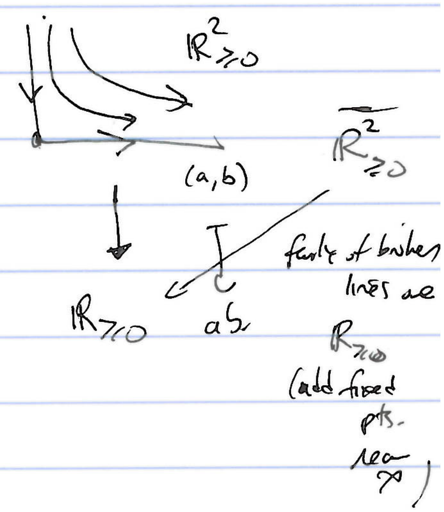
(so, locally finitely many fixed pts)

Construction: Need to make maps $\sigma_n : \mathbb{R}^n_{\geq 0} \longrightarrow \text{Broken}$.

e.g., $S \xrightarrow{f} \mathbb{R}^n_{\geq 0}$, $S \dashrightarrow \tilde{S}$, given f, construct $\tilde{S}$.

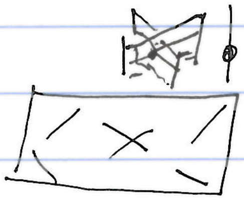
$n=1$: Observe: that $\mathbb{R}^2_{\geq 0}$ has an $\mathbb{R}$-action

$$t(a,b) \longmapsto (e^t a, e^{-t} b),$$



So, given $f : S \longrightarrow \mathbb{R}_{\geq 0}$,

$$\sigma_1 : f \longmapsto f^*\left(\begin{array}{c} \overline{\mathbb{R}}^2_{\geq 0} \\ \downarrow \\ \mathbb{R}_{\geq 0} \end{array} \right).$$



$S \xrightarrow{f} \mathbb{R}_{\geq 0}$

(Broken's topology: analogy of:

$\mathbb{Z}^{op}_{\geq 0}$ w/ Alexandroff ~~closed~~ topology

open/closed sets are upwards closed subsets)

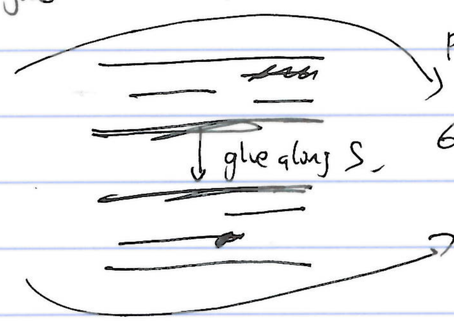
in fact $\mathbb{Z}^{op}_{\geq 0} = |\text{Broken}|$ coarse module.

For σ_n generally, glue the $\sigma_1 f_i$ for each $i = 1, \dots, n$. (in order).

pushout.

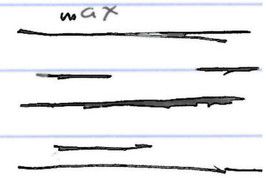
$S \xrightarrow{\text{bottom (min)}} \sigma_1 f_1$

$S \xrightarrow{\text{top (max)}} \sigma_1 f_2$



glue along S. $\sigma_2(f_1, f_2)$ so now

max



min

Theorem: $\mathrm{Shv}^{\otimes}(\text{Broken}; \mathcal{C}^{\otimes}) \simeq A_\infty \mathrm{Alg}^{nu}(\mathcal{C}^{\otimes})$

$\longrightarrow$

What's "factorizable?"

$m: \text{Broken} \times \text{Broken} \xrightarrow{\text{concat}} \text{Broken}$

$\mathcal{F} \boxtimes \mathcal{F} \xleftarrow{\sim} m^* \mathcal{F}$ $\mathcal{F}$

need ↑ & higher & higher equivalences (to encode associativity, etc.)

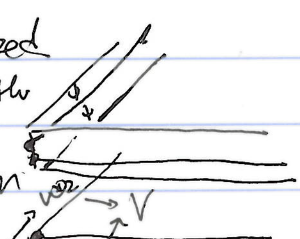
E.g.) $\mathbb{R}_{\geq 0} \xrightarrow{\sigma_2} \text{Broken}$. say $\mathcal{F}$ factorized

Then, $\sigma_2^* \mathcal{F}$ $V \otimes V \xrightarrow{\text{res}} V$

apriori (no unit).

$[1] \to [0]$ (. (estrt)) big open set

Further



from $V^{\otimes 2} \to V$

$V^{\otimes 3} \to V^{\otimes 2}$ $\mathbb{R}^2_{\geq 0}$ $\xrightarrow[\sigma_2]{m \otimes 1}$ Broken.

$[1] \to [0]$

$\uparrow \quad \uparrow$

$[2] \to [1]$

witnesses associativity

Doing this, you'll get a functor:

obj [n], mor surjections $[n] \twoheadrightarrow [m]$

$$\Delta_{surj} \longrightarrow \mathcal{C}^{\otimes}$$

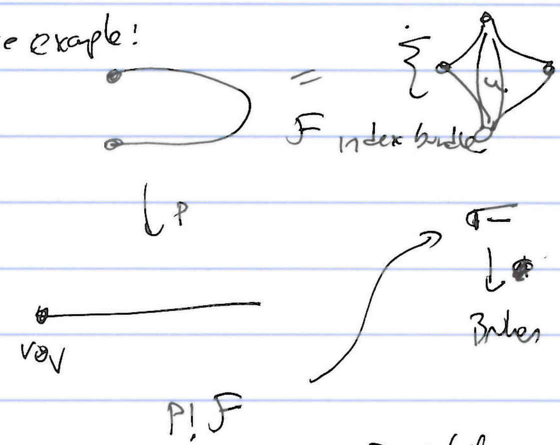
monoidal str. called concatenate.

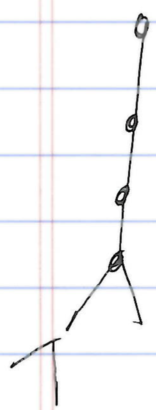
$$[n] \longmapsto V^{\otimes(n-1)}$$

& general point: monoidal strs. like this encode non-unital A_∞ structures.

To actually prove: look at adjoint & counit, & show $\simeq$ $\Rightarrow$ fully faithful.

Actual moduli space example!





$\rightsquigarrow$ will give "colored planar operad!"

Def(things w/ no str) = A_∞ alg.

Def(things w/ A_∞ str) = E_2.