C. Teleman, <u>The Quantum GIT conjecture</u>

~~Dan Pomerleano~~

X compact symplectic, G cpct Lie $\curvearrowright X$ Hamiltonian action

know $QH^\bullet(X)$ + ?? $\rightsquigarrow$ compute ~~$QH^\bullet(X/\!/G)$~~ $QH^\bullet(X::G)$

orbifold version of GIT quotient if action locally free

Fano case: should have a clean answer.

Model: Batyrev's computation of $QH^\bullet$ of toric Fanos

$$QH^\bullet(F) = H^\bullet(BT)/(s_i - q_i)$$

symplectic reduction parameters

Seidel operators, basis of $\pi_1(T)$.

$F = V/\!/T$. $\quad SH^*_T(V)$

Slogan: Fano case, $QH^\bullet(X::G) = QH^*_{LG}(X)$

(?) General case: $= $ direct summand of a deformation of

(key issue: how to identify the summand?)

<u>Ingredients</u>: Wehrheim-Woodward work on Lag'n correspondences, namely

$$\begin{array}{ccc} \mu^{-1}(0) & \hookrightarrow & X \\ \downarrow & & \\ X::G & & \end{array}$$

is a Lag'n in $(X::G)^- \times X$

ought to define functors on Fukaya categories

(yes/known in monotone case, w/o Maslov 2 disks).

general monotone case (w/ Maslov 2 disk) — obstructed by multiples of Id $\longrightarrow$ no effect on QH^*.

Unrestricted version (?) ~~answered by~~ [Fukaya]

Buys you (after work): a bimodule between QH^*'s.

Would like to know it induces an isomorphism.

Interlude: Equivariant Fukaya categories

Can define: objects G-inv't Lag'ns,

Morphisms $HF^*_G(L_1, L_2)$.

(A_∞ operations $\rightsquigarrow$ technical work required).

Claim: should call this the "semi-strict fixed point category"

$$\mathcal{F}(X)^{ssG}$$

"fixed on objects, & homotopy fixed morphisms"

(this is neither the strict nor derived fixed point category; in particular,
In particular, don't expect to compute it from mirror if known).

(~~Ro... there~~

In general, there is a functor (from this) to $\mathcal{F}(X)^{hG}$. the "homotopy fixed point category"

? Quasi-equivalence? (would be <u>great</u>/<u>important</u> if ~~so~~ true in some cases).

Example: $\overset{X}{\mathbb{P}^1}$, $\overset{G}{SU(2)}$ action.

Note: there are <u>no</u> invariant Lag'ns, so $\mathcal{F}(X)^{ssG} = 0$.

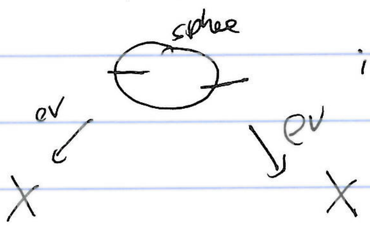
(maybe no problem b/c $Ob(X::G)$ is empty?),

Recall: can deform by $H^*_G(\mathbb{P}^1)$, in particular H^4.

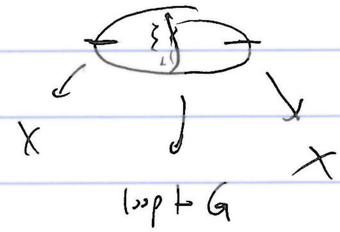
In this case, deformation is generically semisimple w/ infinitely many <u>simples</u>.

This <u>can</u> be computed from the mirror, but not ~~clearly~~ obviously from).

"Quantum" LG action on X.



$id: X \to X$ but also



defines a bundle

study sections of the associated X-bundle.

Can use $H^*(LG)$ to define operators, extending $H_*(G)$ operations.

This works in algebraic geometry [Woodward, Woodward-Gonzalez]

<u>2nd variant in Floer theory:</u>

$g \rightsquigarrow CF^*(X, g)$ fixed pt. Floer coh.

is a bundle of complexes over G

- equivariant under conjugation
- multiplicative: $CF^*(g) \otimes CF^*(h) \to CF^*(gh)$.

- has a 'flat' connection (in the sense of complexes),

of Hamiltonians identifies the fibers at 1 & g if you choose a path.

Equivalently: there is a homomorphism / action of the simplicial group

$$\Omega G \quad \text{on} \quad QH^*(X)$$

which is G-equivariant, and algebraically gives

$$C_* \Omega G \xrightarrow{E_2} QH^*(X)$$ (Remark: being E_2 is a structure, not just a condition)

On $\pi_0 \Omega G = \pi_1 G$, this recovers the Seidel ~~representation of~~ homomorphism.

In general, equivariant homology

$$C_*^G(\Omega G) \quad \text{acts on} \quad QH_G^*(X)$$

E_3 algebra $\qquad$ E_2 algebra over it.

If $G = T$: this is $\mathbb{C}[T^* T^\vee]$

(& these conditions) $\Rightarrow$ "Spec of $QH_G^*(X)$ has Lag'n support in ")

Fact: $QH_G^*(X)$ has "Lag'n support." -- follows from presentation of E_3 & E_2 str. in terms of Gerstenhaber bracket.

E.g., mirror (of projective space) $\mathbb{P}^n$ is functions on

$$(\mathbb{C}^*)^{n+1} \longrightarrow \mathbb{C}^*$$
$$T_q \longrightarrow q$$

dual to tori which act on $\mathbb{C}^{n+1}$

$$W = \{ z_0 + \cdots + z_n \mid z_0 \cdots z_n = q \}$$

relinfo: $\Gamma_{dW} \subset T^* T_q$ (hol. Lag'n)

$\Rightarrow \Gamma_*^G(G; \mathcal{F}^*)$ is an algebra, space of states for the Gauged GW theory

Reminder: G finite, have a twisted sector decomposition: $\leftarrow HF^*(X; g)$

$$QH^*(X :: G) = \Big[\bigoplus_{g \in G} QH^*(X^g) \Big]$$

$\leftarrow$ corrections to multiplications from normal bundle.

X, G, μ: use μ^2 as Morse fcn.,
to define $X^\circ \hookrightarrow X$ semi stable locus

$$\begin{array}{ccc} & & X^\circ \hookrightarrow X \\ G \hookrightarrow \mu^{-1}(0) & \nearrow & \downarrow \\ & \searrow & X::G \end{array}$$

(Guillemin–Sternberg): explain that a disc nhood of $\mu^{-1}(0)$ can be identified with a disc nhood in T^*G bundle over $X::G$.

Assume some miracles: try to restrict

Rmk: X_0 not convex!

- $QH^*_G(X) \xrightarrow{?\sim}$ "$\cancel{SH^*_G(X^\circ)}$" Rmk 2: without deformation! (in Fano case only!

$\cancel{\text{|}\!\wr ?}$

bundle of such things over G, "$SH^*_G(X^\circ_{flat})$" {Bourgeois–Oancea?}

really use T^*G instead of disc nhood of G.

affine fibrations & their SH^* assuming a hol.-sphere in base

Really want: LG-$\overset{equi}{\text{invariant}}$ $SH^*(X^\circ)$ (kills fibre, preserves base)

$QH^*_G(X) \overset{?}{=} QH^*(X::G)$

Doesn't work: but instead use μ^2 to get rid of the unstable locus

Instead:

replace by $HF^*(X; K\cdot\mu^2)$ for very large K (to throw away $X_{unstable}$)

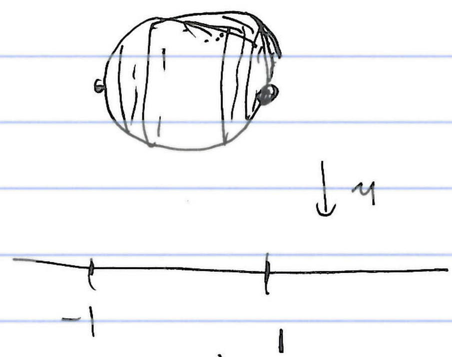
[~~Possible pre-theorem~~] ~~in the Fano~~

Rmk: in the Fano case, QH^* is filtered increasingly by degree

[Pre-theorem]: As K is large, a large part of $CF^*_G(X; K\cdot\mu^2)$ can be identified with $CF^*_G(X^\circ_{flat}; K\cdot p^2)$ (& uniformly over G).

on T^*G.

Example: S^1 acting on $\mathbb{P}^1$



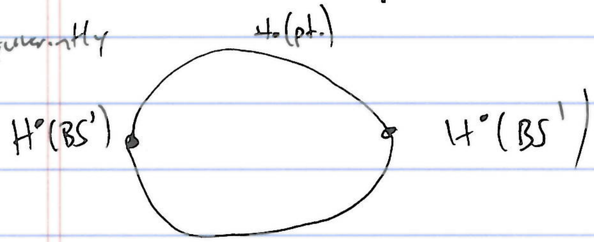
Hamilt flow for $k \cdot \mu^2$

$2k \cdot \mu \cdot H_\mu$

time 1 orbits at $t = \frac{1}{2k\mu}$

$+ \mu^{-1}(0)$

equivariantly



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want to say, as increase k, really exhaust interior things, things: ...

Fano case: degree calculation makes this work ↗.