D. Treumann, Constructible sheaves and immersed Lagrangians, or also

"HOMFLY homology of an immersed Lagrangian"

T^*M a cotangent bundle. Given coordinates $x_1, x_2, \ldots$ coords. on M
$p_1, p_2, \ldots$ coords. dual to $x_1, \ldots, x_n$, &

$L \hookrightarrow T^*M$ (eventually $\looparrowright$) is called "exact" if $\exists\ f: L \to \mathbb{R}$ s.t.
$p_1 dx_1 + p_2 dx_2 + \cdots = df$.

Rough statement of Jin–T.:
If $L \hookrightarrow T^*M$ is embedded, there is a fully faithful functor

$$\left\{ \begin{matrix} \text{(twisted)} \\ \text{local system} \\ \text{on } L \end{matrix} \right\} \hookrightarrow \left\{ \begin{matrix} \text{sheaves on} \\ M \end{matrix} \right\} =: Sh(M)$$

For comparison, "by definition" an exact Lagrangian in T^*M determines a fully faithful functor

$$\left\{ \begin{matrix} \text{local systems} \\ \text{on} \\ L \end{matrix} \right\} \hookrightarrow Fuk(T^*M)$$

Lagrangian HF^* happens here.

Construction of Jin–T. does not use ideas from Gromov. Does use ideas from Arnold.

Def! [Arnold]: $L \hookrightarrow T^*M$ exact Lagn, then its wave front is the map

$$L \longrightarrow \underbrace{M}_{x_1, \ldots, x_n} \times \mathbb{R}$$

$x_1, \ldots, x_n$, $f = \int p\,dx$, "action coordinate". This wave front map determines

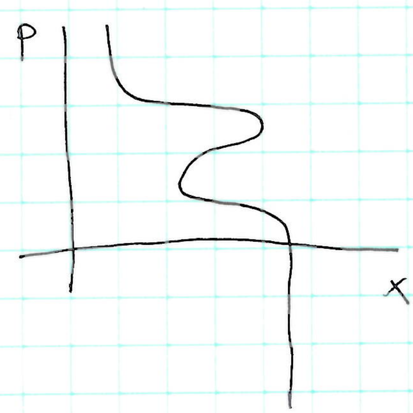
$$L \longrightarrow T^*M$$

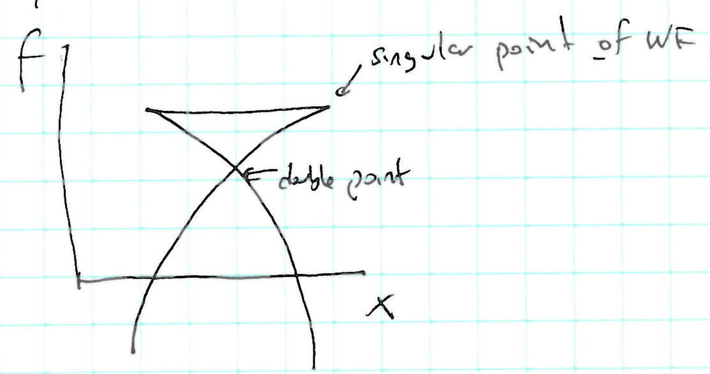
b/c $p_i = \frac{\partial f}{\partial x_i}$.

$WF \subset M \times \mathbb{R}$ image of

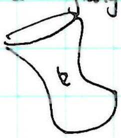
If L is in general position, $L \to T^*M$ is determined by $WF \subset M \times \mathbb{R}$.

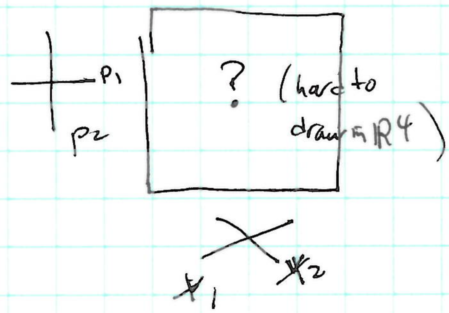
Ex:

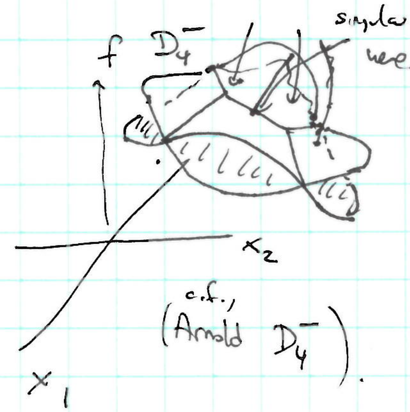




Ex: There's a Lagrangian torus

$L =$  $\subset \mathbb{R}^4$.

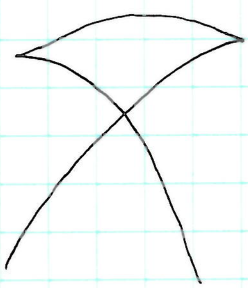




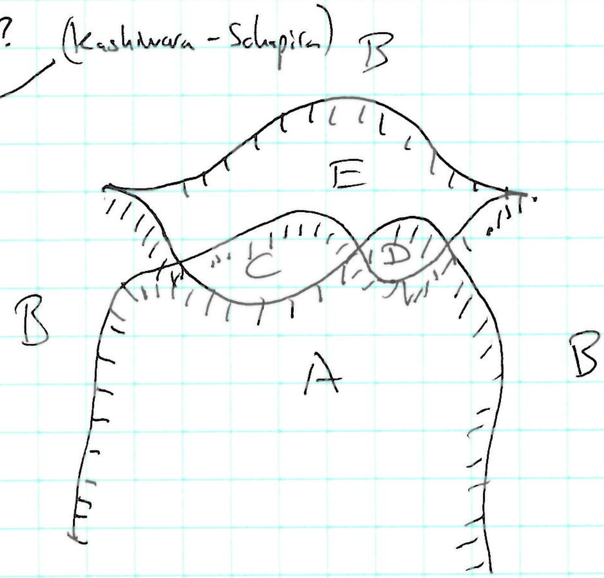
What Xin and I do:

Study $L \hookrightarrow T^*M$ by studying sheaves on $M \times \mathbb{R}$ with "singular support in the wavefront." (rather, in the conic Lag'n lift of L).

What's a sheaf? What's singular support? (Kashiwara–Schapira)

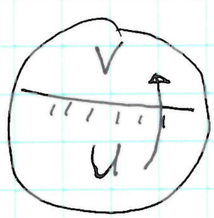


not quite complicated enough.



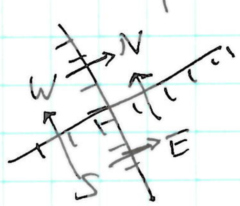
WF ~~decompose~~ chops $M \times \mathbb{R}$ into chambers. First part of sheaf data is a spectrum (or ch-cplx.) in each chamber. (really, a local system of spectra — so if ch. cplxes not contractible, maybe $\exists$ monodromy)

Second part of sheaf data:



is a map of spectra at each smooth point of WF.

At a double point:



impose the "crossing condition":

This square commutes & is exact

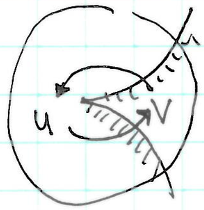
$\Rightarrow \mathrm{Cone}(S \to W) \cong \mathrm{Cone}(E \to N)$

by the way.

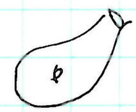
(e.g., $S = \mathrm{fib}(W \times E \to N)$ & $N = \mathrm{cofib}(S \to W \times E)$ in stable ∞-cats.)

At a singular point, impose a different condition:

In 1-D:



$U \to V \to U = 1_U$ i.e., U is a direct summand of V.

At some singular pts, e.g., for  at D_4^-, no condition.

(e.g., condition imposed by D_4^- is empty.)

~~condition~~ (in higher dimensions, ∞'ly many singularities, so hard to combinatorially describe in general).

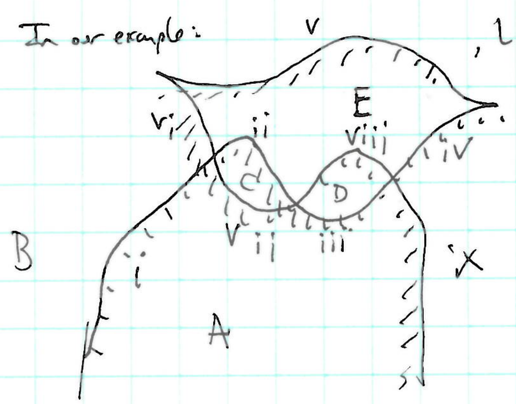
Sheaves w/ singular support in a wavefront WF make a full stable subcat. of $Sh(M \times \mathbb{R})$

$$\cup$$

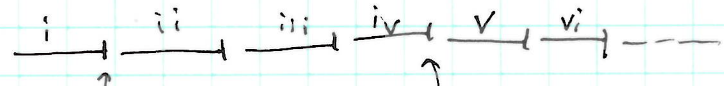
$$Sh_{WF}(M \times \mathbb{R}).$$

Microlocal monodromy:

This is a symbol μmon that is trying to be a functor $Sh_{WF}(M \times \mathbb{R}) \longrightarrow$ Local systems on L.

In our example:  , L is split into arcs:

Cone$(A \to B)$ "B/A" $\cong$ "E/C" $\cong$ "D/A" "F/B" "B/E" $\cong \Sigma E/B$



crossing, induces $\cong$. "Monodromy maps"

cross a cusp, $B/E \cong \Sigma E/B$ by above discussion of maps.

(in other words, monodromy induces an isomorphism up to some twist depending on path taken).

(so, generally, get a functor to twisted local systems!)

Fundamental diagram in $J_{\mathbb{R}}$-T.:

$$\begin{array}{ccc} & \boxed{Sh^{\ominus}_{WF}(M \times \mathbb{R})} & \\ \swarrow \mu mon & & \searrow \text{restriction to } M \times \{-1000000\} \\ \text{Local systems on } L \text{ (maybe twisted)} & & Sh(M) \end{array}$$

$\ominus$: vanishes when "sufficiently high" in action

"asymptotically cptly supported" (or something close).

Properties: Hamiltonian isotopies of T^*M induce an ~~eq~~ htpy equivalence of diagrams (~~GKS~~ Guillermou - Kashiwara - Schapira).

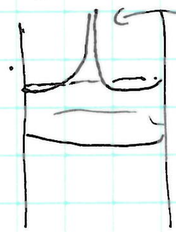
Thm (Ju–T.)

(a) If $L \hookrightarrow T^*M$ is an embedding, then left leg $\mathcal{U}_{mon}$ is an equivalence (generalizing Guillermou case of L cpt.)

(b) If $\partial_\infty L \hookrightarrow T^\infty M$ is an embedding, then right leg is fully faithful functor

(combining these two, get $\mathrm{Loc}\, L \hookrightarrow Sh(M)$

(what can go wrong?



self-intersection at ∞).

• If L is compact, then right arrow $Sh^0_{wp}(M \times \mathbb{R}) \longrightarrow Sh(M)$

$$\begin{tikzcd} Sh^0_{wp}(M\times\mathbb{R}) \arrow[r] \arrow[dr, squiggly] & Sh(M) \\ & \mathrm{Loc}(M) \arrow[u] \end{tikzcd}$$

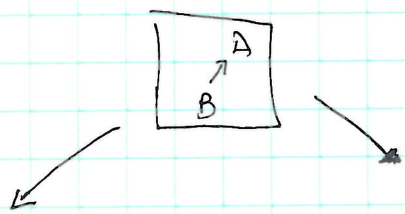
(immersed!)

Ex: $M = $ point, $L = 2$ points. (need to also choose $f|_L$; answer will vary based on this).

co-oriented WF & sheaf?

& left arrow

Loc L is not an equivalence!



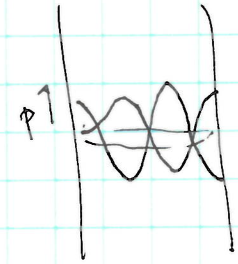
B.

$\left(A, \frac{A}{B}\right)$

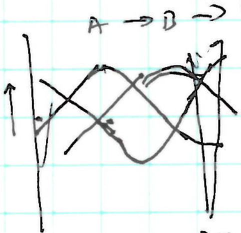
so $\mathcal{U}_{mon}^{-1}$ is a "multivalued functor" sending $\left(A, \frac{A}{B}\right)$ to all extensions of " $A \to B \to A/B$.

Ex: $M = S^1$,

$L = S^1 \xrightarrow{\text{exact}} T^*M$ e.g., $p\uparrow$



& WF: $f\uparrow$

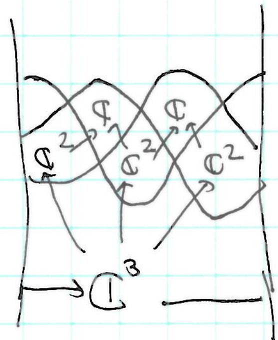


0 2π

V. Shende, T. E. Zaslow: think of WF as a leg. knot, studied these:

Result is: a category containing a moduli space which is a "Bott-Samelson resolution of GL_n"

e.g.:



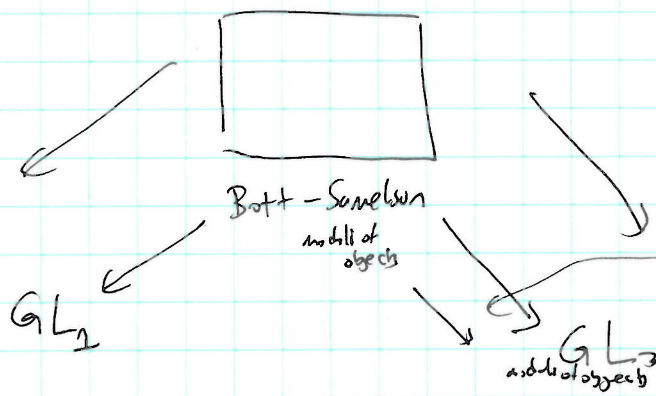
Bott-Samelson resolution of

GL_3 in this case

variety/manifold:

(some space of maps between flags, + monodromy)

So, in this example:



c.f. Representation theory

Lusztig "Horocycle correspondence"

look at second page of "Leray SS" of map (doubly graded) + weight filtration (3rd grading)

Thm [Webster-Williamson]: This diagram (in this case) knows the triply graded Khovanov-Rozansky or "HOMFLY" homology of the knot you get by taking generic braid of embedding in 3-sphere.

works for "Braid positive knots".

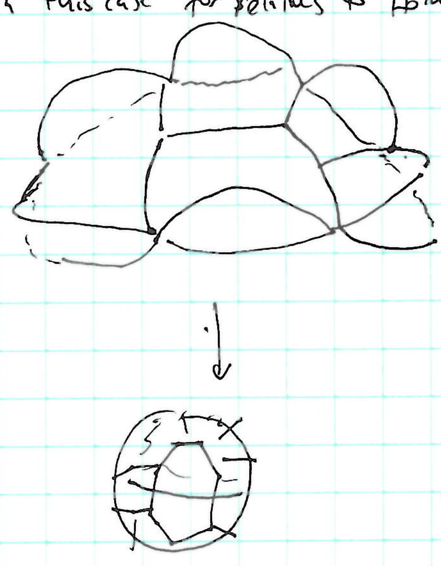
c.f. also work of D. Rutherford in this case for relations to HOMFLY polynomial.

One more picture:

Ex: $M = S^2$

L = genus g surface

(Shende-T-Zaslow)



unresolved WF

In this case:

the exact Lagn has $g+3$ double pts,

$2g+2$ vertices

$3g+3$ edges

$g+3$ faces.

& diagram

moduli of colorings of "coloring in graph on S^2"

$\mathrm{Log}(\Sigma_g)$

μ_{mon} is an embedding

constant (or not quite)

(b/c no local system on S^2, at least underived).

or rather: GL_2 (which acts on space of colorings).

Running through HOMFLY homology:

get: H^0(space of colorings, + weight grading) spread out in 3 degrees in some way.

Rmk: $\exists$ Floer theory for immersed Lag's, & it looks very different from this.

Q: how to compare them?