Math 113 Homework 1

Due Friday, April 12th, 2013 by 4 pm

Please remember to write down your name and Stanford ID number, and to staple your solutions. Solutions are due to the Course Assistant, Graham White, in his office, 380-380R (either hand your solutions directly to him or leave the solutions under his door).

Book problems: Solve Axler Chapter 1 problems 3, 4, 7, 14, 15 (pages 19-20).

1. Let X be any set, $\mathbb{F}$ any field, and W a vector space over $\mathbb{F}$. Denote by

$$W^X \tag{0.1}$$

the set of functions from X to W. As a special case of this, if $\mathbb{F} = \mathbb{R}$, $X = [0, 1]$, and $W = \mathbb{R}$, this construction gives

$$\mathbb{R}^{[0,1]} = Fun([0, 1], \mathbb{R}), \tag{0.2}$$

the set of functions from $[0, 1]$ to $\mathbb{R}$ that we described in class.

Define an addition on W^X by

$$(f + g)(x) = f(x) + g(x)$$

using the vector space addition in W, and define scalar multiplcation on W^X, for a scalar $a \in \mathbb{F}$, by

$$(a \cdot f)(x) = a \cdot f(x)$$

using the scalar multiplication on W.

In class, we (almost) proved that $\mathbb{R}^{[0,1]} = Fun([0, 1], \mathbb{R})$ is a vector space over $\mathbb{R}$. Prove more generally that with this addition and scalar multiplication, W^X is a vector space over $\mathbb{F}$.

2. The set $\mathbb{N}$ is defined to consist of the *natural numbers*:

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, \ldots\}.$$

Let U denote the set of functions

$$U = \{f : \mathbb{N} \longrightarrow \mathbb{F} \mid f(2i) = 2f(i) \text{ for } i \in \mathbb{N}\}.$$

Is U is a subspace of $\mathbb{F}^{\mathbb{N}}$ (see Problem 1 for a definition of $\mathbb{F}^{\mathbb{N}}$)? Prove or disprove.

3. (a) Prove that the only subspaces of $\mathbb{F}^1$ are $\{\mathbf{0}\}$ and $\mathbb{F}^1$.

(b) Consider the set

$$U = \{(x_1, x_2, x_3) \in \mathbb{F}^3 \mid x_1 x_2 x_3 = 0\}.$$

Is U a subspace of $\mathbb{F}^3$? Prove or disprove.

4. Let V be a vector space (over some field $\mathbb{F}$), and $U_1, \ldots, U_m$ subspaces. Prove that the sum

$$U_1 + \cdots + U_m$$

is the smallest subspace of V containing each subspace U_i. More specifically, prove that if W is any subspace which contains U_i for $i = 1, \ldots, m$, then W contains the sum $U_1 + \cdots + U_m$.

5. Let $U_1 = \{(a, -a, 0) \mid a \in \mathbb{F}\}$, $U_2 = \{(b, 0, -b) \mid b \in \mathbb{F}\}$ and $U_3 = \{(-c, -c, 2c) \mid c \in \mathbb{F}\}$. These are all subspaces of $\mathbb{F}^3$ (you may assume this without proof).

(a) Describe the subspace $U_1+U_2+U_3$ (and of course, justify this description with proof).

(b) Let $W = U_1 + U_2 + U_3$. Is W the direct sum of U_1, U_2, and U_3? Prove or disprove.