Math 113 — Homework 1

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Book problems:

3. Let v be an arbitrary element of the vector space V. By the definition of the additive inverse, we have that

$$-(-v) + (-v) = 0$$

and that

$$v + (-v) = 0.$$

Therefore,

$$\begin{aligned} -(-v) + (-v) &= v + (-v) && \text{(equality is transitive)} \\ (-(-v) + (-v)) + v &= (v + (-v)) + v && \text{(adding } v \text{ to both sides)} \\ -(-v) + ((-v) + v) &= (v + (-v)) + v && \text{(addition is associative)} \\ -(-v) + 0 &= 0 + v && (v + -v = -v + v = 0) \\ -(-v) &= v && (u + 0 = 0 + u = u) \end{aligned}$$

as required.

4. Let $a \in \mathbf{F}$ and $v \in V$ with $av = 0$. This equation is saying that when you multiply the vector v by the scalar a, the result is the zero vector. To show that either $a = 0$ or that $v = 0$, we just need to rule out the case where both a and v are nonzero. It suffices to show that if $a \neq 0$ then $v = 0$. (Keep in mind that we are comparing a to the zero scalar, and v to the zero vector).

Assume that $a \neq 0$. Then there is some $a^{-1} \in \mathbf{F}$ with $a^{-1}a = 1$. Hence

$$\begin{aligned} av &= 0 && \\ a^{-1}(av) &= a^{-1}0 && \text{(multiplying both sides by the scalar } a^{-1}) \\ (a^{-1}a)v &= a^{-1}0 && \text{(scalar multiplication is associative)} \\ (a^{-1}a)v &= 0 && (b0 = 0 \text{ for any } b \in \mathbf{F}) \\ (1)v &= 0 && (a^{-1}a = 1) \\ v &= 0 && (1v = v) \end{aligned}$$

We have shown that if $a \neq 0$, then $v = 0$. Therefore, either $a = 0$ or $v = 0$, as required.

7. Let U be the union of the two lines $x = 0$ and $y = 0$. This set is closed under scalar multiplication, as $a(x, 0) = (ax, 0)$ and $a(0, y) = (0, ay)$. The set U contains $(1, 0)$ and $(0, 1)$, but not their sum $(1, 1)$, so U is not closed under addition. Therefore U is not a subspace.

Indeed, any nonempty set U which is closed under scalar multiplication must contain 0, as it is closed under multiplication by the scalar 0, and $0v = 0$ for any $v \in V$. Therefore the only way that U could not be a subspace is for it to not be closed under addition.

Any multiplicatively closed set in $\mathbb{R}^2$ is the union of lines through the origin. The only such sets which are subspaces are a single line, or all the lines. (And interpreting the union of no lines as just the origin, that's a subspace too).

14. Let W be the subset of $\mathcal{P}(\mathbf{F})$ defined by

$$W = \{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x_1 + a_0 \mid a_5 = a_2 = 0\}.$$

The set W contains, for example, $p(x) = x$, so it is nonempty. The coefficients of x^2 and x^5 in the sum of two polynomials are the sums of the respective coefficients, so W is closed under addition (as $0 + 0 = 0$). The coefficients of x^2 and x^5 in the product of a polynomial and a constant are the coefficients in the original polynomial multiplied by that constant, so W is closed under scalar multiplication (as $k0 = 0$ for any $k \in \mathbf{F}$). Therefore W is a subspace of $\mathcal{P}(\mathbf{F})$.

To show that $\mathcal{P}(\mathbf{F}) = U \oplus W$, we need to check that $\mathcal{P}(\mathbf{F}) = U + W$ and $U \cap W = \{0\}$, by Proposition 1.9.

Any polynomial can be written as (the sum of its x^2 and x^5 terms) plus (all the other terms). These are in U and W respectively, so $\mathcal{P}(\mathbf{F}) = U + W$. Any polynomial in U is of the form $ax^2 + bx^5$, and if it's in W then $a = b = 0$, as a polynomial can be written in only one way as the sum of scalar multiples of $1, x, x^2, \ldots$. Therefore a polynomial in $U \cap W$ is the zero polynomial, so $U \cap W = \{0\}$. (We need not check that the zero polynomial is actually in U and in V. It must be, because they are subspaces).

So $\mathcal{P}(\mathbf{F}) = U \oplus W$, as required.

15. The statement is false. Consider $V = \mathbb{R}^2$ as a vector space over the field $\mathbb{R}$. Let U_1, U_2, W be defined as follows:

$$U_1 = \left\{ \begin{pmatrix} a \\ a \end{pmatrix}, a \in \mathbb{R} \right\}, U_2 = \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix}, a \in \mathbb{R} \right\}, W = \left\{ \begin{pmatrix} 0 \\ a \end{pmatrix}, a \in \mathbb{R} \right\}.$$

Each of these is comprised of all real multiples of a single vector, so it is easy to check that they are subspaces. Any element of $V = \mathbb{R}^2$ can be written as follows:

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix} + \begin{pmatrix} 0 \\ b - a \end{pmatrix} \in U_1 + W.$$

Or

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix} \in U_2 + W.$$

Therefore $V = U_1 + W = U_2 + W$.

Now, any vector $\begin{pmatrix} a \\ b \end{pmatrix}$ in $U_1 \cap W$ satisfies $a = b$ and $a = 0$, so is the zero vector. Thus $U_1 \cap W = \{0\}$. Likewise, if $\begin{pmatrix} a \\ b \end{pmatrix} \in U_2 \cap W$, then $a = 0$ and $b = 0$, so $U_2 \cap W = \{0\}$.

By Proposition 1.9, $V = U_1 \oplus W = U_2 \oplus W$. But $U_1 \neq U_2$, so this is a counterexample to the claim.

Other problems:

1. Elements of W^X are functions from the set X to the vector space W. We will see that the vector space axioms for W^X will follow from those for the vector space W.

To show that two functions are equal, we must show that they are equal on each element x of X. Let f, g and h be arbitrary elements of W^X, a and b a arbitrary elements of $\mathbb{F}$, and x any element of X. We calculate the following:

- Associativity of addition:

$$\begin{aligned}
((f+g)+h)(x) &= (f+g)(x)+h(x) && \text{(definition of addition in } W^X\text{)}\\
&= (f(x)+g(x))+h(x) && \text{(definition of addition in } W^X\text{)}\\
&= f(x)+(g(x)+h(x)) && \text{(addition in } W \text{ is associative)}\\
&= f(x)+(g+h)(x) && \text{(addition in } W^X\text{)}\\
&= (f+(g+h))(x) && \text{(addition in } W^X\text{)}
\end{aligned}$$

  We have shown that $((f+g)+h)(x) = (f+(g+h))(x)$ for each $x \in X$, so $(f+g)+h = f+(g+h)$

- Commutativity of addition:

$$\begin{aligned}
(f+g)(x) &= f(x)+g(x) && \text{(definition of addition in } W^X\text{)}\\
&= g(x)+f(x) && \text{(addition in } W \text{ is commutative)}\\
&= (g+f)(x) && \text{(addition in } W^X\text{)}
\end{aligned}$$

  We have shown that $(f+g)(x) = (g+f)(x)$ for each $x \in X$, so $(f+g) = (g+f)$

- Additive identity:

  Let $\mathbf{0}$ be the function from X to W defined by $\mathbf{0}(x) = 0$ for each $x \in X$.

  Then for each $x \in X$,

$$(f+\mathbf{0})(x) = f(x)+\mathbf{0}(x) = f(x)+0 = f(x).$$

  Likewise, $(\mathbf{0}+f)(x) = f(x)$. Thus $f+\mathbf{0} = \mathbf{0}+f = f$, so $\mathbf{0}$ is an additive identity for W^X.

- Additive inverses:

  Define the function $-f$ by $(-f)(x) = -f(x)$ (taking the additive inverse in W).

  Then for each $x \in X$,

$$(f+(-f))(x) = f(x)+(-f)(x) = f(x)+-f(x) = 0 = \mathbf{0}(x).$$

  Likewise, $((-f)+f)(x) = \mathbf{0}(x)$. Thus $f+(-f) = (-f)+f = \mathbf{0}$, so $-f$ is an additive inverse for f.

- Distributivity over vector addition:

$$\begin{aligned}
(a(f+g))(x) &= a((f+g)(x)) && \text{(definition of scalar multiplication in } W^X\text{)}\\
&= a(f(x)+g(x)) && \text{(addition in } W^X\text{)}\\
&= a(f(x))+a(g(x)) && \text{(distributivity in } W\text{)}\\
&= (af)(x)+(ag)(x) && \text{(scalar multiplication in } W^X\text{)}
\end{aligned}$$

  We have shown that $(a(f+g))(x) = (af+ag)(x)$ for each $x \in X$, so $a(f+g) = af+ag$.

- Distributivity over scalar addition:

$$\begin{aligned}
((a+b)f)(x) &= (a+b)(f(x)) && \text{(definition of scalar multiplication in } W^X\text{)}\\
&= a(f(x))+b(f(x)) && \text{(distributivity in } W\text{)}\\
&= (af)(x)+(bf)(x) && \text{(scalar multiplication in } W^X\text{)}\\
&= (af+bf)(x) && \text{(addition in } W^X\text{)}
\end{aligned}$$

  We have shown that $((a+b)(f))(x) = (af+bf)(x)$ for each $x \in X$, so $(a+b)(f) = af+bf$.

- Compatibility of multiplication:

$$\begin{aligned}(a(b(f)))(x) &= a((b(f))(x)) && \text{(definition of scalar multiplication in } W^X\text{)}\\ &= a(b(f(x))) && \text{(scalar multiplication in } W^X\text{)}\\ &= (ab)(f(x)) && \text{(compatibility of scalar multiplication in } W\text{)}\\ &= (ab(f))(x) && \text{(scalar multiplication in } W^X\text{)}\end{aligned}$$

  We have shown that $(a(b(f)))(x) = (ab(f))(x)$ for each $x \in X$, so $a(b(f)) = (ab)(f)$.

- Scalar multiplication by the identity:

  Let 1 be the multiplicative identity of $\mathbb{F}$. For any f,

$$(1f)(x) = 1(f(x)) = f(x),$$

  because W is a vector space over $\mathbb{F} \ni 1$. Hence $1f = f$.

We have checked each of the vector space axioms, so W^X is a vector space over $\mathbb{F}$.

2. The set U is a subspace of $\mathbb{F}^{\mathbb{N}}$. We check that it is nonempty, closed under addition, and closed under scalar multiplication.

   The function $f(i) = 0, i \in \mathbb{N}$ is in U, so U is nonempty.

   Let f and g be arbitrary functions in U, and let $h = f + g$. Then for any $i \in \mathbb{N}$, we have that:

$$\begin{aligned}h(2i) &= f(2i) + g(2i) && \text{(definition of addition in } \mathbb{F}^{\mathbb{N}}\text{)}\\ &= 2f(i) + 2g(i) && (f \text{ and } g \text{ are in } U)\\ &= 2h(i) && \text{(definition of addition in } \mathbb{F}^{\mathbb{N}}\text{)}\end{aligned}$$

   Therefore $h = f + g$ is an element of U, so U is closed under addition.

   Likewise, let f be an arbitrary element of $\mathbb{F}^{\mathbb{N}}$ and a any scalar. Let $h = af$. Then for any $i \in \mathbb{N}$, we have that:

$$\begin{aligned}h(2i) &= af(2i) && \text{(definition of scalar multiplication in } \mathbb{F}^{\mathbb{N}}\text{)}\\ &= 2af(i) && (f \text{ is in } U)\\ &= 2h(i) && \text{(definition of scalar multiplication in } \mathbb{F}^{\mathbb{N}}\text{)}\end{aligned}$$

   Therefore $h = af$ is an element of U, so U is closed under scalar multiplication.

   The subset U is nonempty, closed under addition, and closed under scalar multiplication, so is a subspace of $\mathbb{F}^{\mathbb{N}}$.

3. (a) To disambiguate elements of $\mathbb{F}$ and elements of $\mathbb{F}^1$, we will use parentheses to denote elements of $\mathbb{F}^1$, as these are vectors of length 1 whose entries are in $\mathbb{F}$. That is, a represents an element of $\mathbb{F}$, while (a) represents an element of $\mathbb{F}^1$. Let U be a subspace of $\mathbb{F}^1$. Any subspace contains (0). If U does not contain any other elements of $\mathbb{F}^1$, then $U = \{(0)\}$. If it does, then let (a) be a nonzero element of U, where a is a nonzero element of $\mathbb{F}$.

   Any nonzero element of a field has a multiplicative inverse, so there exists $a^{-1} \in \mathbb{F}$ with $a^{-1}a = 1$. For any element of $\mathbb{F}^1$, $(b), b \in \mathbb{F}$, consider the scalar ba^{-1}. The subspace U is closed under scalar multiplication, so $ba^{-1}(a) = (b)$ is an element of U. Therefore any element of $\mathbb{F}^1$ is in U, so $U = \mathbb{F}^1$. Hence U is either (0) or $\mathbb{F}^1$, as required.

   (b) The set U is not a subspace of $\mathbb{F}^3$. The elements $(1, 1, 0)$ and $(0, 0, 1)$ of $\mathbb{F}^3$ are in U, but their sum, $(1, 1, 1)$, is not. Therefore U is not closed under addition. (Using that $0x = 0$ and that $1x = x$ for any $x \in \mathbb{F}$, and that $0 \neq 1$ in $\mathbb{F}$).

4. Let $u_1, u_2, \ldots, u_m$ be arbitrary elements of $U_1, U_2, \ldots, U_m$, so that $u_1 + u_2 + \ldots + u_m$ is an arbitary element of $U_1 + U_2 + \ldots + U_m$. If W is a subspace of V containing each U_i, then it contains each u_i. If W is a subspace, then it is closed under addition, so it contains $u_1 + u_2$, and $(u_1 + u_2) + u_3$, and $((u_1 + u_2) + u_3) + u_4$, etc. Therefore W contains $u_1 + u_2 + \ldots + u_m$. But this was an arbitary element of $U_1 + U_2 + \ldots + U_m$, so W contains all of $U_1 + U_2 + \ldots + U_m$.

5. (a) We claim that $U_1 + U_2 + U_3$ is the set of points (x, y, z) with $x + y + z = 0$. Let this set be W. The set W is a subspace, because it contains $(0, 0, 0)$, if $x_1 + y_1 + z_1 = 0$ and $x_2 + y_2 + z_2 = 0$ then

$$(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = 0,$$

and if $x_1 + y_1 + z_1 = 0$ then $ax_1 + ay_1 + az_1 = 0$.

The subspace W contains U_1, U_2 and U_3, by comparing an element of any of these subspaces to the equation defining W. Therefore W contains $U_1 + U_2 + U_3$, as W is closed under addition.

Let (x, y, z) be an arbitrary element of W. Then $z = -x - y$, so

$$\begin{aligned}(x, y, z) &= (x, y, -x - y) \\ &= (x, -x, 0) + (0, x + y, -x - y) + (0, 0, 0) \in U_1 + U_2 + U_3\end{aligned}$$

Therefore W is contained in $U_1 + U_2 + U_3$.

We have shown that $W \subseteq U_1 + U_2 + U_3 \subseteq W$, so $W = U_1 + U_2 + U_3$.

(b) Assume that W is the direct sum $(U_1 \oplus U_2) \oplus U_3$, where we may parenthesise the associative direct sum arbitrarily. Then the intersection of $(U_1 \oplus U_2)$ and U_3 must contain only the zero vector. But $(1, -1, 0) + (-2, 0, 2) = (-1, -1, 2)$ is in both $(U_1 \oplus U_2)$ and U_3, and $1 \neq 0$ in $\mathbb{F}$, so the intersection of $(U_1 \oplus U_2)$ and U_3 is not just the zero vector, a contradiction.

Therefore W is not the direct sum $U_1 \oplus U_2 \oplus U_3$.