Math 113 Homework 2

Due Friday, April 19th, 2013 by 4 pm

Please remember to write down your name and Stanford ID number, and to staple your solutions. Solutions are due to the Course Assistant, Graham White, in his office, 380-380R (either hand your solutions directly to him or leave the solutions under his door). As usual, please justify all of your solutions and/or answers with carefully written proofs.

Book problems: Solve Axler Chapter 2 problems 1, 3, 5, 8, 12, 14, 16 (pages 35-36). Read if necessary the definitions of $\mathbb{F}^\infty$ and $\mathcal{P}_m(\mathbb{F})$ in Axler first.

1. (*The linear dual of a vector space*) Recall that if X is a set and W is a vector space, then W^X is the set of all functions from X to W. You proved on Homework 1 that W^X is a vector space.

Now, if V is a vector space over the field $\mathbb{F}$, the **(linear) dual vector space** V^* is the set of all functions $f : V \to \mathbb{F}$ that satisfy

$$f(\mathbf{v}+\mathbf{w}) = f(\mathbf{v}) + f(\mathbf{w}) \text{ for all } \mathbf{v}, \mathbf{w} \in V$$
$$f(a \cdot \mathbf{v}) = a \cdot f(\mathbf{v}) \text{ for all } \mathbf{v} \in V, a \in \mathbb{F}.$$

(Namely, V^* is the set of all *linear maps* from V to $\mathbb{F}$).

(a) By definition, V^* is a subset of the vector space $\mathbb{F}^V$ of functions from V to $\mathbb{F}$. Prove that V^* is a subspace of $\mathbb{F}^V$, and therefore a vector space itself.

(b) Assume that $\dim V = n$, and that $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is a basis for V. Find a basis for V^*. What is $\dim V^*$?

2. If V and W are vector spaces, let $\mathcal{L}(V, W)$ denote the subset of W^V consisting of functions that are *linear maps*; that is, functions $T : V \to W$ that satisfy

$$T(\mathbf{v}+\mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w});$$
$$T(a \cdot \mathbf{v}) = a \cdot T(\mathbf{v}).$$

(a) Prove that $\mathcal{L}(V, W)$ is a vector space.

(b) Suppose that $\dim(V) = 2$ and $\dim(W) = 3$, and let $(\mathbf{v}_1, \mathbf{v}_2)$ be a basis for V and $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ be a basis for W. Find a basis for $\mathcal{L}(V, W)$. What is $\dim(\mathcal{L}(V, W))$?

3. (*Equivalence relations and quotient spaces*) An **equivalence relation** on a set X is any binary relation on X (a relation between some pairs of elements in X), denoted $\sim$, which satisfies the following properties:

- (*reflexive property*) $x \sim x$ for all elements $x \in X$;
- (*symmetric property*) $x \sim y$ implies $y \sim x$, for any pair of elements $x, y \in X$;

- (*transitive property*) If $x \sim y$ and $y \sim z$, then $x \sim z$, for any triple of elements $x, y, z \in X$.

One simple example of an equivalence relation is *equality*: it is clear that $x = x$, $x = y$ if $y = x$, and if $x = y$ and $y = z$ then $x = z$. In this exercise, we examine an important equivalence relation arising in linear algebra.

(a) Let V be a vector space and $W \subset V$ a subspace. Define a relation on the set V as follows:

$$\mathbf{u} \sim \mathbf{v} \text{ if } \mathbf{u} - \mathbf{v} \in W. \tag{1}$$

Prove that this defines an equivalence relation on V. What is the equivalence class $[\mathbf{0}]$?

(b) Given a set X and an equivalence relation $\sim$ on X, one can partition X into a collection of **equivalence classes**. An equivalence class is a subset of X consisting of all elements that are similar to a given element. Any element $a \in X$ belongs to a single equivalence class, called $[a]$:

$$[a] := \{x \in X \mid x \sim a\}.$$

Two elements a and b have the same equivalence class ($[a] = [b]$) if and only if $a \sim b$ (a and b are both called *representatives* of the equivalence class). The set of distinct equivalence classes is denoted

$$X/\sim \; := \; \{[a] \mid a \in X\}.$$

Now, return to the situation from part (a), with V a vector space, W a subspace, and $\sim$ as defined as in (1). Define the **quotient space** of V by W to be the set of equivalence classes under $\sim$:

$$V/W := V/\sim .$$

Elements of V/W are equivalence classes of elements in V, denoted either by $[\mathbf{v}]$ or $\mathbf{v} + W$. This second notation is because as a set,

$$[\mathbf{v}] = \mathbf{v} + W := \{\mathbf{v} + \mathbf{w} \mid \mathbf{w} \in W\}.$$

Prove that V/W is a vector space.

(*Hint*: first, define a sum and scalar multiplication on V/W. For the case of the sum: you might try to define the sum on V/W as $[\mathbf{v}] + [\mathbf{w}] := [\mathbf{v} + \mathbf{w}]$ for some representatives $\mathbf{v}$, $\mathbf{w}$, but you do not a priori know this is well-defined! To check that it is, you must prove that if you picked a different set of elements $\mathbf{v}'$ and $\mathbf{w}'$ in the same equivalence classes as $\mathbf{v}$ and $\mathbf{w}$, then the sum $\mathbf{v}' + \mathbf{w}'$ is in the same equivalence class as $\mathbf{v} + \mathbf{w}$. Finally you must check the properties of a vector space).

(c) Suppose that V and W are both finite-dimensional. Prove that $\dim(V/W) = \dim(V) - \dim(W)$.

(*Hint*: Start by extending a basis

$$(\mathbf{w}_1, \ldots, \mathbf{w}_m)$$

of W to a basis

$$(\mathbf{w}_1, \ldots, \mathbf{w}_m, \mathbf{u}_1, \ldots, \mathbf{u}_{m-n})$$

of V. Then, show that $(\mathbf{u}_1 + W, \ldots, \mathbf{u}_{m-n} + W)$ is a basis for V/W.)

Remark: In class, we showed that there always exists a complementary subspace to W, i.e. a subspace U with $W \oplus U = V$. However, this subspace is not-unique (its construction depended on a choice of an extension of a basis of W to a basis of V). In contrast, the quotient V/W, which has the same dimension as U, is a space that does not depend on a choice of basis (we say that it is **canonical**). This comes at only one expense: V/W is no longer a subspace of V.

(d) Consider the special case $V = \mathbb{R}^2$, and $W = \mathrm{span}(1, 1)$. Give a geometric description of the quotient space V/W.

4. What is the dimension of the vector space spanned by

(a) The vectors $(1, i, 1+i, 0)$, $(-i, 1, 1-i, 0)$, and $(1-i, 1+i, 2, 0)$ in $\mathbb{C}^4$?

(b) The functions $f(x) = \sin(x)$, $\sin(x+\pi/3)$, $\sin(x+\pi/6)$ in $\mathcal{C}^0(\mathbb{R}, \mathbb{R})$? (Recall: $\mathcal{C}^0(\mathbb{R}, \mathbb{R})$ is the vector space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$.)

5. Let $U \subset \mathbb{R}^\infty$ be the subspace of sequences $(v_1, v_2, \ldots, v_i, \ldots)$ satisfying

$$v_{i+2} = v_i - v_{i+1} \text{ for all } i.$$

(you do not need to prove U is a subspace for homework, but you should check it for yourself). Prove in fact that U is finite-dimensional with $\dim(U) = 2$. (More specifically, find a basis of length 2 of U.) Finally, find a *complementary* subspace; that is, a subspace $W \subset \mathbb{R}^\infty$ such that

$$U \oplus W = \mathbb{R}^\infty.$$

(Note: W will not be finite-dimensional!)