Math 113 — Homework 2

Graham White

April 20, 2013

Book problems:

1. Consider an arbitrary element v of V. The set $\{v_1, v_2, \ldots v_n\}$ spans V, so there are scalars $a_1, a_2, \ldots, a_n$ with

$$v = a_1v_1 + a_2v_2 + \ldots + a_nv_n.$$

Rearranging this equation, we have that

$$v = a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) + \ldots + (a_1 + a_2 + \ldots + a_n)v_n. \qquad (1)$$

Hence

$$v \in \text{Span}(v_1, (v_2 - v_1), (v_3 - v_2), \ldots, (v_{n-1} - v_n), v_n).$$

But v was an arbitrary element of V, so the set $\{v_1, (v_2 - v_1), (v_3 - v_2), \ldots, (v_{n-1} - v_n), v_n\}$ spans V, as required. (To verify equation 1, calculate the coefficient of each v_i).

3. Assume that $(v_1 + w, v_2 + w, \ldots, v_n + w)$ is linearly dependent. Then there are scalars $a_1, a_2, \ldots, a_n$, not all zero, with

$$0 = a_1(v_1 + w) + a_2(v_2 + w) + \ldots + a_n(v_n + w).$$

Hence

$$-(a_1 + a_2 + \ldots + a_n)w = a_1v_1 + a_2v_2 + \ldots + a_nv_n.$$

If $(a_1+a_2+\ldots+a_n) = 0$, then $a_1v_1+a_2v_2+\ldots+a_nv_n = 0$, so all of the a_i are 0, because the set $\{v_1, v_2, \ldots v_n\}$ is linearly independent. But this contradicts the definition of $a_1, a_2, \ldots, a_n$, so $(a_1 + a_2 + \ldots + a_n) \neq 0$, hence we may divide by the scalar $(a_1 + a_2 + \ldots + a_n)$. Therefore

$$w = \frac{-a_1}{a_1 + a_2 + \ldots + a_n}v_1 + \ldots + \frac{-a_n}{a_1 + a_2 + \ldots + a_n}v_n,$$

so w is in the span of $\{v_1, v_2, \ldots v_n\}$, as required.

5. For each positive integer i, let e_i be the sequence with ith element 1 and each other element 0. For any n, we will show that $e_1, e_2, \ldots, e_n$ are linearly independent. Assume that

$$a_1e_1 + a_2e_2 + \ldots a_ne_n = 0.$$

Using the definition of the e_is, we have that

$$(a_1, a_2, \ldots, a_n) = (0, 0, \ldots, 0).$$

Therefore $a_1 = a_2 = \ldots = a_n = 0$. We have shown that $e_1, e_2, \ldots, e_n$ are linearly independent for any fixed n. Therefore for any n, the dimension of $\mathbf{F}^\infty$ is at least n. Hence $\mathbf{F}^\infty$ is infinite dimensional.

8. Define the following vectors:

$$v_1 = (3, 1, 0, 0, 0)$$
$$v_2 = (0, 0, 7, 1, 0)$$
$$v_3 = (0, 0, 0, 0, 1)$$

By the definition of U, each of these vectors is in U. Let $av_1 + bv_2 + cv_3$ be an arbitrary linear combination of v_1, v_2 and v_3. If $av_1 + bv_2 + cv_3 = 0$, then

$$(3a, a, 7b, b, c) = (0, 0, 0, 0, 0),$$

by the definitions of v_1, v_2 and v_3. Comparing the coordinates of these vectors, $a = b = c = 0$. Therefore v_1, v_2 and v_3 are linearly independent.

Consider an arbitrary element $x = (a, b, c, d, e)$ of U. By the definition of U, we have that $a = 3b$ and $c = 7d$. Therefore $x = (3b, b, 7d, d, e) = bv_1 + dv_2 + ev_3$. Therefore x is a linear combination of v_1, v_2 and v_3, so $\{v_1, v_2, v_3\}$ spans U.

We have that $\{v_1, v_2, v_3\}$ is linearly independent and spans U, so it is a basis of U.

12. For any nonnegative integer n, consider the vector space $P_n(\mathbf{F})$, and the functions $\{1, x, x^2, \ldots, x^n\}$ in this space. This set spans $P_n(\mathbf{F})$, because any polynomial is a sum of monomials, and is linearly independent, because two polynomials are equal only if all of their coefficients are equal. Thus the set is a basis, so the dimension of $P_n(\mathbf{F})$ is $n + 1$, for any n.

    Now, consider $p_0, p_1, \ldots, p_m \in P_m(\mathbf{F})$, with $p_i(2) = 0$ for each i. Then for each i, we may write $p_i(x) = (x - 2)g_i(x)$, where g_i is an element of $P_{m-1}(\mathbf{F})$. The polynomials $g_0, g_1, \ldots, g_m$ are $m + 1$ elements of $P_{m-1}(\mathbf{F})$, which we have shown is m–dimensional. Therefore they are linearly dependent, so there exist some scalars $a_0, a_1, \ldots, a_m$ such that

    $$a_0g_0 + a_1g_1 + \ldots + a_mg_m = 0.$$

    Multiplying by $x - 2$, we get that

    $$a_0f_0 + a_1f_1 + \ldots + a_mf_m = 0,$$

    so the polynomials $f_0, f_1, \ldots, f_m$ are linearly dependent in $P_m(\mathbf{F})$, as required.

    Another approach to this problem would be to use the fact that if $p_0, p_1, \ldots, p_m \in P_m(\mathbf{F})$ are linearly independent in a vector space of dimension $m + 1$, then they form a basis. But any linear combination of the p_i will be zero at the point 2, so the collection of p_i can't span $P_m(\mathbf{F})$, because $P_m(\mathbf{F})$ contains polynomials which are not zero at the point 2, like $p(x) = x$. (This is just an outline — the various statements here would, of course, require proof).

14. From Theorem 2.18 of Axler, we have that

    $$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

    The space $U + W$ is a subspace of $\mathbb{R}^9$, so has dimension at most 9. We are given that each of U and W have dimension 5. Hence we have that

    $$5 + 5 - \dim(U \cap W) \leq 9$$

    Therefore $\dim(U \cap W) \geq 1$. The dimension of $\{0\}$ is zero, so $U \cap W \neq \{0\}$.

16. If V is finite dimensional and each U_i is a subspace of V, then each U_i is finite dimensional, by Proposition 2.7 of Axler. For each i, let A_i be a basis of U_i (A_i is a set of vectors). For each i, the set A_i has $\dim(U_i)$ elements.

    The sets A_i span the subspaces U_i, so the set $A_1 \cup A_2 \cup \ldots \cup A_m$ spans the subspace $U_1 + U_2 + \ldots + U_m$. Therefore

    $$\begin{aligned}\dim(U_1 + U_2 + \ldots + U_m) &\leq |A_1 \cup A_2 \cup \ldots \cup A_m| && \text{(Size of any spanning set is at least the dimension)}\\ &\leq |A_1| + |A_2| + \ldots + |A_m|\\ &= \dim(U_1) + \dim(U_2) + \ldots + \dim(U_m)\end{aligned}$$

    This proves the required inequality.

Other problems:

1. (a) To prove that V^* is a subspace of $\mathbb{F}^V$, we need to show that V^* is nonempty, closed under addition, and closed under scalar multiplication.

In the following, let u and v be arbitrary elements of V and a be an arbitrary element of $\mathbb{F}$.

Let $\mathbf{0}$ be the zero function from V to $\mathbb{F}$. Then

$$\mathbf{0}(u+v) = 0 = 0 + 0 = \mathbf{0}(u) + \mathbf{0}(v)$$

and

$$\mathbf{0}(av) = 0 = a0 = a\mathbf{0}(v).$$

Therefore $\mathbf{0}$ is a linear function, so is contained in V^*.

Let f and g be arbitrary elements of V^*. Then

$$\begin{aligned}
(f+g)(u+v) &= f(u+v) + g(u+v) && \text{(Definition of } (f+g)) \\
&= f(u) + f(v) + g(u) + g(v) && (f \text{ and } g \text{ are linear}) \\
&= f(u) + g(u) + f(v) + g(v) && \text{(Addition in } \mathbb{F} \text{ is commutative)} \\
&= (f+g)(u) + (f+g)(v) && \text{(Definition of } (f+g))
\end{aligned}$$

and

$$\begin{aligned}
(f+g)(av) &= f(av) + g(av) && \text{(Definition of } (f+g)) \\
&= a(f(v)) + a(g(v)) && (f \text{ and } g \text{ are linear}) \\
&= a(f(v) + g(v)) && \text{(Distributivity in } \mathbb{F}) \\
&= a((f+g)(v)) && \text{(Definition of } (f+g))
\end{aligned}$$

Therefore $f+g$ is a linear function, so V^* is closed under addition.

Let f be an arbitrary element of V^*, and λ be any element of $\mathbb{F}$.

$$\begin{aligned}
(\lambda f)(u+v) &= \lambda(f(u+v)) && \text{(Definition of } \lambda f) \\
&= \lambda(f(u) + f(v)) && (f \text{ is linear}) \\
&= \lambda(f(u)) + \lambda(f(v)) && \text{(Distributivity in } \mathbb{F}) \\
&= (\lambda f)(u) + (\lambda f)(v) && \text{(Definition of } \lambda f)
\end{aligned}$$

and

$$\begin{aligned}
(\lambda f)(av) &= \lambda(f(av)) && \text{(Definition of } \lambda f) \\
&= \lambda(a(f(v))) && (f \text{ is linear}) \\
&= a(\lambda(f(v))) && \text{(Multiplication in } \mathbb{F} \text{ is commutative)} \\
&= a((\lambda f)(v)) && \text{(Definition of } \lambda f)
\end{aligned}$$

Therefore λf is a linear function, so V^* is closed under scalar multiplication.

The subset V^* of $\mathbb{F}^V$ is nonempty, closed under addition, and closed under scalar multiplication, so it is a subspace of $\mathbb{F}^V$, as required.

(b) Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V. For each i, define $f_i : V \longrightarrow \mathbb{F}$ by

$$f_i(a_1v_1 + a_2v_2 + \ldots + a_nv_n) = a_i.$$

These functions are well-defined because $\{v_1, v_2, \ldots, v_n\}$ is a basis for V, so each element v of V can be uniquely expressed as $v = a_1v_1 + a_2v_2 + \ldots + a_nv_n$, for some $a_1, a_2, \ldots, a_n \in \mathbb{F}$.

We need to show that each f_i is a linear map. Let $u = a_1v_1 + \ldots + a_nv_n$ and $v = b_1v_1 + \ldots + b_nv_n$ be any elements of V, and λ be any element of $\mathbb{F}$. Then

$$\begin{aligned}
f_i(u+v) &= f_i(a_1v_1 + \ldots + a_nv_n + b_1v_1 + \ldots + b_nv_n) && \text{(Definition of } u \text{ and } v\text{)}\\
&= f_i(a_1v_1 + b_1v_1 + \ldots + a_nv_n + b_nv_n && \text{(Commutativity of addition in } \mathbb{F}^V\text{)}\\
&= f_i((a_1+b_1)v_1 + \ldots + (a_n+b_n)v_n) && \text{(Distributivity of scalar multiplication in } \mathbb{F}^V\text{)}\\
&= a_i + b_i && \text{(Definition of } f_i\text{)}\\
&= f_i(a_1v_1 + \ldots + a_nv_n) + f_i(b_1v_1 + \ldots + b_nv_n) && \text{(Definition of } f_i\text{)}\\
&= f_i(u) + f_i(v) \quad \text{(Definition of } u \text{ and } v\text{)}
\end{aligned}$$

and

$$\begin{aligned}
f_i(\lambda v) &= f_i(\lambda(a_1v_1 + \ldots + a_nv_n) && \text{(Definition of } u\text{)}\\
&= f_i((\lambda a_1)v_1 + \ldots + (\lambda a_n)v_n) && \text{(Distributivity of scalar multiplication in } \mathbb{F}^V\text{)}\\
&= \lambda a_i && \text{(Definition of } f_i\text{)}\\
&= \lambda f_i(a_1v_1 + \ldots + a_nv_n) && \text{(Definition of } f_i\text{)}\\
&= \lambda f_i(u) && \text{(Definition of } u\text{)}
\end{aligned}$$

Therefore each f_i is a linear map. We will now show that $\{f_1, f_2, \ldots, f_n\}$ is a basis for V^*. To check that this set is linearly independent, assume that $a_1f_1 + a_2f_2 + \ldots + a_nf_n = \mathbf{0}$, for some scalars $a_1, a_2, \ldots, a_n$. Then for each i,

$$\begin{aligned}
0 &= \mathbf{0}(v_i)\\
&= (a_1f_1 + a_2f_2 + \ldots + a_nf_n)(v_i)\\
&= a_1f_1(v_i) + \ldots + a_nf_n(v_i) && \text{(Addition and scalar multiplication in } \mathbb{F}^V\text{)}\\
&= a_if_i(v_i) && (f_j(v_i) = 0 \text{ for } i \neq j)\\
&= a_i \cdot 1 && (f_i(v_i) = 1)\\
&= a_i
\end{aligned}$$

Therefore, if $a_1f_1 + a_2f_2 + \ldots + a_nf_n = \mathbf{0}$, then each $a_i = 0$, so the set $\{f_1, f_2, \ldots, f_n\}$ is linearly independent.

Now, let f be any element of V^*. Define $b_i = f(v_i)$ for each i. Let $v = a_1v_1 + \ldots + a_nv_n$ be any element of V. Then

$$\begin{aligned}
f(v) &= f(a_1v_1 + a_2v_2 + \ldots + a_nv_n)\\
&= f(a_1v_1) + f(a_2v_2) + \ldots + f(a_nv_n) && (f \text{ is linear})\\
&= a_1f(v_1) + a_2f(v_2) + \ldots + a_nf(v_n) && (f \text{ is linear})\\
&= a_1b_1 + a_2b_2 + \ldots + a_nb_n && \text{(Definition of } b_i\text{)}\\
&= b_1a_1 + b_2a_2 + \ldots + b_na_n && \text{(Multiplication in } \mathbb{F} \text{ is commutative)}\\
&= b_1f_1(a_1v_1 + \ldots + a_nv_n) + \ldots + b_nf_n(a_1v_1 + \ldots + a_nv_n) && \text{(Definition of } f_i\text{)}\\
&= b_1f_1(v) + \ldots + b_nf_n(v) && \text{(Definition of } v\text{)}\\
&= (b_1f_1 + \ldots + b_nf_n)(v) && \text{(Addition and scalar multiplication in } \mathbb{F}^W\text{)}
\end{aligned}$$

Therefore for each $v \in V$, we have that $f(v) = (b_1 f_1 + \ldots + b_n f_n)(v)$, so $f = (b_1 f_1 + \ldots + b_n f_n)$. We have shown that any $f \in V^*$ can be written as a linear combination of $\{f_1, f_2, \ldots, f_n\}$, so this set spans V^*.

We know that $\{f_1, f_2, \ldots, f_n\}$ is linearly independent and spans V^*, so it is a basis of V^*.

The space V^* has a basis with n elements, so $\dim(V^*) = n$.

2. (a) For this question, we need to assume that V and W are vector spaces over the same field, $\mathbb{F}$. From the first homework set, we know that W^V is a vector space, so it suffices to show that the subset $\mathcal{L}(V, W)$ is a subspace. That is, that $\mathcal{L}(V, W)$ is nonempty, closed under addition, and closed under scalar multiplication. The proof of this is very similar to the proof in the previous question that V^* is a subspace of $\mathbb{F}^V$.

In the following, let u and v be arbitrary elements of V and a be an arbitrary element of $\mathbb{F}$.

Let **0** be the zero function from V to W. That is, the constant function taking the value 0_W, the additive identity of W. Then

$$\mathbf{0}(u+v) = 0_W = 0_W + 0_W = \mathbf{0}(u) + \mathbf{0}(v)$$

and

$$\mathbf{0}(av) = 0_W = a0_W = a\mathbf{0}(v).$$

Therefore **0** is a linear function, so is contained in $\mathcal{L}(V, W)$.

Let f and g be arbitrary elements of $\mathcal{L}(V, W)$. Then

$$\begin{aligned}
(f+g)(u+v) &= f(u+v) + g(u+v) && \text{(Definition of } (f+g)\text{)} \\
&= f(u) + f(v) + g(u) + g(v) && (f \text{ and } g \text{ are linear}) \\
&= f(u) + g(u) + f(v) + g(v) && \text{(Addition in } W \text{ is commutative)} \\
&= (f+g)(u) + (f+g)(v) && \text{(Definition of } (f+g)\text{)}
\end{aligned}$$

and

$$\begin{aligned}
(f+g)(av) &= f(av) + g(av) && \text{(Definition of } (f+g)\text{)} \\
&= a(f(v)) + a(g(v)) && (f \text{ and } g \text{ are linear}) \\
&= a(f(v) + g(v)) && \text{(Distributivity in } W\text{)} \\
&= a((f+g)(v)) && \text{(Definition of } (f+g)\text{)}
\end{aligned}$$

Therefore $f + g$ is a linear function, so $\mathcal{L}(V, W)$ is closed under addition.

Let f be an arbitrary element of $\mathcal{L}(V, W)$, and λ be any element of $\mathbb{F}$.

$$\begin{aligned}
(\lambda f)(u+v) &= \lambda(f(u+v)) && \text{(Definition of } \lambda f\text{)} \\
&= \lambda(f(u) + f(v)) && (f \text{ is linear}) \\
&= \lambda(f(u)) + \lambda(f(v)) && \text{(Distributivity in } W\text{)} \\
&= (\lambda f)(u) + (\lambda f)(v) && \text{(Definition of } \lambda f\text{)}
\end{aligned}$$

and

$$
\begin{aligned}
(\lambda f)(av) &= \lambda(f(av)) && \text{(Definition of } \lambda f\text{)} \\
&= \lambda(a(f(v))) && (f \text{ is linear}) \\
&= (\lambda a)(f(v)) && \text{(Distributivity in } W\text{)} \\
&= (a\lambda)(f(v)) && \text{(Multiplication in } \mathbb{F} \text{ is commutative)} \\
&= a(\lambda(f(v))) && \text{(Distributivity in } W\text{)} \\
&= a((\lambda f)(v)) && \text{(Definition of } \lambda f\text{)}
\end{aligned}
$$

Therefore λf is a linear function, so $\mathcal{L}(V, W)$ is closed under scalar multiplication.

The subset $\mathcal{L}(V, W)$ of W^V is nonempty, closed under addition, and closed under scalar multiplication, so it is a subspace of W^V. Hence $\mathcal{L}(V, W)$ is a vector space.

(b) For each pair (i, j) with $1 \leq i \leq 2$ and $1 \leq j \leq 3$, we define a function $f_{ij} : V \longrightarrow W$ by $f_{ij}(a_1v_1 + a_2v_2) = a_iw_j$. These functions are well-defined because any element of V can be uniquely written as a linear combination of a_1 and a_2, as $\{a_1, a_2\}$ is a basis of V.

We will show that the six functions f_{ij} form a basis for $\mathcal{L}(V, W)$.

To show that the functions f_{ij} are linearly independent, assume that some linear combination of them is the zero function,

$$a_{11}f_{11} + a_{12}f_{12} + a_{13}f_{13} + a_{21}f_{21} + a_{22}f_{22} + a_{23}f_{23} = \mathbf{0}$$

Then for each $i \in \{1, 2\}$, we have that

$$
\begin{aligned}
0 &= \mathbf{0}(v_i) \\
&= (a_{11}f_{11} + a_{12}f_{12} + a_{13}f_{13} + a_{21}f_{21} + a_{22}f_{22} + a_{23}f_{23})(v_i) \\
&= a_{11}f_{11}(v_i) + a_{12}f_{12}(v_i) + a_{13}f_{13}(v_i) + a_{21}f_{21}(v_i) + a_{22}f_{22}(v_i) + a_{23}f_{23}(v_i) && \text{(Operations in } W^V\text{)} \\
&= a_{i1}w_1 + a_{i2}w_2 + a_{i3}w_3 && \text{(Definition of } f_{ij}\text{)}
\end{aligned}
$$

Now, $\{w_1, w_2, w_3\}$ is a basis for W, so is linearly independent. Therefore if $a_{i1}w_1 + a_{i2}w_2 + a_{i3}w_3 = 0$, then $a_{i1} = a_{i2} = a_{i3} = 0$. So if

$$a_{11}f_{11} + a_{12}f_{12} + a_{13}f_{13} + a_{21}f_{21} + a_{22}f_{22} + a_{23}f_{23} = \mathbf{0},$$

then each $a_{ij} = 0$. Therefore the set $\{f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}\}$ is linearly independent.

To show that the functions f_{ij} span $\mathcal{L}(V, W)$, consider any linear function f from V to W. We will show that f is a linear combination of the f_{ij}.

Because $\{w_1, w_2, w_3\}$ is a basis of W, we have that $f(v_1) = b_{11}w_1 + b_{12}w_2 + b_{13}w_3$ and $f(v_2) = b_{21}w_1 + b_{22}w_2 + b_{23}w_3$ for some scalars $b_{ij}, i \in \{1, 2\}, j \in \{1, 2, 3\}$. Let $v = a_1v_1 + a_2v_2$ be an arbitrary element of V. Then we have

$$
\begin{aligned}
f(v) &= f(a_1v_1 + a_2v_2) \\
&= a_1f(v_1) + a_2f(v_2) && (f \text{ is linear}) \\
&= a_1(b_{11}w_1 + b_{12}w_2 + b_{13}w_3) + a_2(b_{21}w_1 + b_{22}w_2 + b_{23}w_3) && \text{(Definitions of } b_{ij}\text{)} \\
&= b_{11}a_1w_1 + b_{12}a_1w_2 + b_{13}a_1w_3 + b_{21}a_2w_1 + b_{22}a_2w_2 + b_{23}a_2w_3 && \text{(Commutativity in } \mathbb{F}\text{)} \\
&= b_{11}f_{11}(a_1v_1 + a_2v_2) + \ldots + b_{23}f_{23}(a_1v_1 + a_2v_2) && \text{(Definition of the } f_{ij}\text{)} \\
&= b_{11}f_{11}(v) + b_{12}f_{12}(v) + b_{13}f_{13}(v) + b_{21}f_{21}(v) + b_{22}f_{22}(v) + b_{23}f_{23}(v) && \text{(Definition of } v\text{)} \\
&= (b_{11}f_{11} + b_{12}f_{12} + b_{13}f_{13} + b_{21}f_{21} + b_{22}f_{22} + b_{23}f_{23})(v) && \text{(Operations in } W^V\text{)}
\end{aligned}
$$

Therefore for each $v \in V$, we have that

$$f(v) = (b_{11}f_{11} + b_{12}f_{12} + b_{13}f_{13} + b_{21}f_{21} + b_{22}f_{22} + b_{23}f_{23})(v),$$

so

$$f = b_{11}f_{11} + b_{12}f_{12} + b_{13}f_{13} + b_{21}f_{21} + b_{22}f_{22} + b_{23}f_{23}.$$

Therefore f is a linear combination of the f_{ij}, so the set $\{f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}\}$ spans $\mathcal{L}(V, W)$.

We have shown that $\{f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}\}$ is linearly independent and spans $\mathcal{L}(V, W)$, so it is a basis for $\mathcal{L}(V, W)$.

The space $\mathcal{L}(V, W)$ has a basis of length 6, so $\dim(\mathcal{L}(V, W)) = 6$.

3. (a) To show that $\sim$ is an equivalence relation, we need to show that it is reflexive, symmetric, and transitive.

Let v be any element of V. Then $v - v = 0$, and 0 is in W because W is a subspace of V. Therefore $v \sim v$.

Let v, w be any two elements of V with $v \sim w$. Then $(v - w) \in W$. The space W of V is closed under scalar multiplication, so $-(v-w) = (w-v)$ is an element of W. (You should be familiar enough with the field and vector space axioms to verify manipulations like this one if necessary). If $(w - v)$ is an element of W, then $w \sim v$. We have shown that if $v \sim w$ then $w \sim v$.

Let u, v, w be any three elements of V with $u \sim v$ and $v \sim w$. Then $(u - v)$ and $(v - w)$ are elements of W. But W is a subspace of V, so is closed under addition, so $(u - v) + (v - w) = (u - w)$ is an element of W. Therefore $u \sim w$.

We have shown that $\sim$ is reflexive, symmetric, and transitive, so it is an equivalence relation.

An element v of V is in the equivalence class $[0]$ if and only if $v \sim 0$. But this is equivalent to $v - 0 = v$ being an element of W. Hence the equivalence class $[0]$ is exactly W.

(b) We define addition and scalar multiplication on V/W as follows. Let $[v]$ and $[w]$ be arbitrary elements of V/W and a be any scalar. Define $[v] + [w]$ to be $[v + w]$ and $a[v]$ to be $[av]$. Firstly, we need to show that these operations are well-defined, that is, that the result does not depend on the representatives v and w chosen.

Let v' and w' be any representatives of the equivalence classes $[v]$ and $[w]$. Then $(v' - v) \in W$ and $(w' - w) \in W$. We need to show that the equivalence classes $[v]+[w] = [v+w]$ and $[v']+[w'] = [v'+w']$ are the same. But $(v' + w') - (v + w) = (v' - v) + (w' - w)$, which is an element of W because W is closed under addition. Hence the sum of two equivalence classes does not depend on the representatives chosen.

Let v' be any representative of the equivalence class $[v]$. Then $(v' - v) \in W$. We need to show that the equivalence classes $a[v] = [av]$ and $a[v'] = [av']$ are the same. We have that $av' - av = a(v' - v)$, which is in W because W is closed under scalar multiplication. Therefore the multiple of an equivalence class by a scalar does not depend on the representative.

We have shown that our addition and scalar multiplication of equivalence classes are well defined. Now we need to show that V/W is a vector space over $\mathbb{F}$ with these operations. Let $[u]$, $[v]$ and $[w]$ be arbitrary elements of V/W, and a and b be any scalars.

- Associativity of addition:

$$\begin{aligned} ([u] + [v]) + [w] &= [u + v] + w \\ &= [(u + v) + w] \\ &= [u + (v + w)] && \text{(Addition in } V \text{ is associative)} \\ &= [u] + [v + w] \\ &= [u] + ([v] + [w]) \end{aligned}$$

Therefore addition in V/W is associative.

- Commutativity of addition:

$$\begin{aligned}[u]+[v] &= [u+v] \\ &= [v+u] && \text{(Addition in } V \text{ is commutative)} \\ &= [v]+[u]\end{aligned}$$

  Therefore addition in V/W is commutative.

- Additive identity:

  For each $[v] \in V/W$,
  $$[v]+[0] = [v+0] = [v].$$
  Likewise, $[0]+[v]=[v]$. Thus $[v]+[0]=[0]+[v]=[v]$, so $[0]$ is an additive identity for V/W.

- Additive inverses:

  For each $[v] \in V/W$,
  $$[v]+[-v] = [v+(-v)] = [0].$$
  Likewise, $[-v]+[v]=[0]$. Thus $[v]+[-v]=[-v]+[v]=[0]$, so $[-v]$ is an additive inverse of $[v]$ in V/W. Hence each element of V/W has an additive inverse.

- Distributivity over vector addition:

$$\begin{aligned}a([u]+[v]) &= a([u+v]) \\ &= [a(u+v)] \\ &= [au+av] && \text{(Distributivity in } V\text{)} \\ &= [au]+[av] \\ &= a[u]+a[v]\end{aligned}$$

  Therefore $a([u]+[v]) = a[u]+a[v]$ for each $a, [u]$ and $[v]$.

- Distributivity over scalar addition:

$$\begin{aligned}(a+b)[v] &= [(a+b)v] \\ &= [av+bv] && \text{(Distributivity in } V\text{)} \\ &= [av]+[bv] \\ &= a[v]+b[v]\end{aligned}$$

  We have shown that $(a+b)[v] = a[v]+b[v]$ for each a, b and $[v]$.

- Compatibility of multiplication:

$$\begin{aligned}(a(b[v])) &= a[bv] \\ &= [a(bv)] \\ &= [(ab)v] && \text{(compatibility of scalar multiplication in } V\text{)} \\ &= (ab)[v]\end{aligned}$$

  We have shown that $a(b[v]) = (ab)[v]$ for each a, b and $[v]$.

- Scalar multiplication by the identity:

  Let 1 be the multiplicative identity of $\mathbb{F}$. Then $1[v] = [1v] = [v]$, as $1v = v$ in V.

We have checked each of the vector space axioms, so V/W is a vector space over $\mathbb{F}$, as required.

(c) Let the dimension of V be n and the dimension of W be m. As suggested in the hint, let $\{w_1, w_2, \ldots, w_m\}$ be a basis of W, and extend it to a basis $\{w_1, w_2, \ldots, w_m, u_1, u_2, \ldots, u_{n-m}\}$ of V. We will show that $\{[u_1], [u_2], \ldots, [u_{n-m}]\}$ is a basis of V/W.

Let $[v]$ be any element of V/W. Then v is an element of V, so can be written as

$$v = a_1w_1 + a_2w_2 + \ldots + a_mw_m + b_1u_1 + b_2u_2 + \ldots + b_{n-m}u_{n-m}.$$

Let $w = a_1w_1 + a_2w_2 + \ldots + a_mw_m$. Then $w \in W$, so $v \sim (v - w)$. Therefore

$$\begin{aligned} [v] &= [v - w] \\ &= [b_1u_1 + b_2u_2 + \ldots + b_{n-m}u_{n-m}] \\ &= b_1[u_1] + b_2[u_2] + \ldots + b_{n-m}[u_{n-m}] \end{aligned}$$

Therefore any element $[v]$ of V/W is a linear combination of $\{[u_1], [u_2], \ldots, [u_{n-m}]\}$, so this set spans V/W.

Now, assume that

$$b_1[u_1] + b_2[u_2] + \ldots + b_{n-m}[u_{n-m}] = [0].$$

Then

$$[b_1u_1 + b_2u_2 + \ldots + b_{n-m}u_{n-m}] = [0].$$

We know that $[0] = W$, and $\{w_1, w_2, \ldots, w_m\}$ is a basis of W. Hence there are $a_1, a_2, \ldots, a_m$ such that

$$b_1u_1 + b_2u_2 + \ldots + b_{n-m}u_{n-m} = a_1w_1 + a_2w_2 + \cdots + a_mw_m.$$

This is equivalent to

$$-a_1w_1 - a_2w_2 - \cdots - a_mw_m + b_1u_1 + b_2u_2 + \ldots + b_{n-m}u_{n-m} = 0$$

But $\{w_1, w_2, \ldots, w_m, u_1, u_2, \ldots, u_{n-m}\}$ is a basis of v, so each a_i and each b_i are equal to 0.

We have shown that if

$$b_1[u_1] + b_2[u_2] + \ldots + b_{n-m}[u_{n-m}] = [0]$$

then each b_i is equal to 0, so the set $\{[u_1], [u_2], \ldots, [u_{n-m}]\}$ is linearly independent.

The set $\{[u_1], [u_2], \ldots, [u_{n-m}]\}$ is linearly independent and spans V/W, so it is a basis for V/W.

The space V/W has a basis of length $n - m = \dim(V) - dim(W)$, so

$$\dim(V/W) = \dim(V) - \dim(W).$$

(d) The quotient space V/W is comprised of cosets of W in V. That is, translated copies of W. The elements of V/W are the lines in $\mathbb{R}^2$ of gradient $+1$. Two such lines are added by choosing a point on each line, adding those as usual in $\mathbb{R}^2$, and taking the line of gradient $+1$ through the resulting point. Likewise, a line is multiplied by a scalar by multiplying any point on the line by that scalar and taking the line of gradient $+1$ through the resulting point.

4. (a) Let

$$v_1 = (1, i, 1 + i, 0)$$

$$v_2 = (-i, 1, 1 - i, 0)$$

and

$$v_3 = (1 - i, 1 + i, 2, 0).$$

Note that $v_2 = -i \cdot v_1$ and $v_3 = (1 - i) \cdot v_1$. Therefore

$$\text{Span}(v_1, v_2, v_3) = \text{Span}(v_1).$$

The vector v_1 is nonzero, so $\text{Span}(v_1)$ is one-dimensional. Thus $\{v_1, v_2, v_3\}$ spans a one-dimensional subspace of $\mathbb{C}^4$.

(This result depends on the field of scalars being $\mathbb{C}$. It is also possible to consider $\mathbb{C}^4$ as a vector space over $\mathbb{R}$, in which case the subspace would be two-dimensional. If this was intended, though, it would have been explicitly stated. The vector space $\mathbb{C}^4$ without further qualification is a vector space over $\mathbb{C}$.)

(b) The functions $\sin(x)$ and $\sin(x+\frac{\pi}{3})$ are both nonzero. There are many ways to see that neither is a multiple of the other — for example, only the former is zero at $x = 0$ and only the latter is zero at $x = -\frac{\pi}{3}$. Hence these two functions span a subspace of $\mathcal{C}^0(\mathbb{R}, \mathbb{R})$ that is at least two dimensional.

But for any θ, the function

$$\sin(x + \theta) = \cos(\theta)\sin(x) + \sin(\theta)\cos(x)$$

is a linear combination of $\sin(x)$ and $\cos(x)$. Hence any function $\sin(x + \theta)$ is contained in the two dimensional subspace of $\mathcal{C}^0(\mathbb{R}, \mathbb{R})$ spanned by $\cos(x)$ and $\sin(x)$.

Therefore the subspace of $\mathcal{C}^0(\mathbb{R}, \mathbb{R})$ spanned by $\sin(x)$, $\sin(x + \frac{\pi}{3})$ and $\sin(x + \frac{\pi}{6})$ is exactly the two dimensional subspace of $\mathcal{C}^0(\mathbb{R}, \mathbb{R})$ spanned by $\cos(x)$ and $\sin(x)$.

5. Let a and b be the sequences

$$a = (1, 0, 1, -1, 2, -3, 5, -8, 13, \dots)$$

and

$$b = (0, 1, -1, 2, -3, 5, -8, 13, -21, \dots)$$

defined by

$$(a_1, a_2) = (1, 0) \text{ and } (b_1, b_2) = (0, 1)$$

and satisfying $u_{i+2} = u_i - u_{i+1}$ for each i.

We will show that $\{a, b\}$ is a basis for U. Each of a and b is in U, by definition. They are both nonzero and neither is a multiple of the other, so the set $\{a, b\}$ is linearly independent.

Let $u = (u_1, u_2, u_3, \dots)$ be any element of U. Then $u - u_1 a - u_2 b = (0, 0, (u_3 - u_1 a_3 - u_2 b_3), \dots)$ is an element of U. But by a trivial induction, if the first two terms of a sequence in U are zero, then all of its terms are zero. Therefore $u = u_1 a + u_2 b$, so the set $\{a, b\}$ spans U.

Therefore $\{a, b\}$ is a basis for U, so U is two dimensional.

Let W be the set of sequences of the form

$$w = (0, 0, w_3, w_4, w_5, \dots).$$

It is clear that W is a subspace, because the property that the first two terms are zero is preserved by addition and by scalar multiplication, and there are plenty of sequences with the first two terms zero.

Consider any element of $U \cap W$. This is a sequence whose first two terms are zero, and which satisfies $u_{i+2} = u_i - u_{i+1}$ for each i. As noted above, this must be the zero sequence, so $U \cap W = \{0\}$.

Consider any element $x = (x_1, x_2, x_3, x_4, \dots)$ of $\mathbb{R}^\infty$. Then $(x_1 a + x_2 b)$ is an element of U and $x - (x_1 a + x_2 b)$ is an element of W, so $x \in U + W$. Therefore $U + W = \mathbb{R}^\infty$.

We have shown that $U + W = \mathbb{R}^\infty$ and that $U \cap W = \{0\}$, so $\mathbb{R}^\infty = U \oplus W$.