# Math 113 Homework 3

Due Friday, April 26th, 2013 by 4 pm

Please remember to write down your name and Stanford ID number, and to staple your solutions. Solutions are due to the Course Assistant, Graham White, in his office, 380-380R (either hand your solutions directly to him or leave the solutions under his door). As usual, please justify all of your solutions and/or answers with carefully written proofs.

**Book problems**: Solve Axler Chapter 3 problems 4, 9, 15, 22, 23, 24 (pages 59-62).

**1.** *More on the quotient.* Let $V$ be a vector space and $W \subset V$ a subspace. In class and on the previous Homework (see #3), we defined and developed some properties of the *quotient space*

$$V/W,$$

a vector space defined as the collection of subsets of the form $[\mathbf{v}] = \mathbf{v} + W$, with operations of addition and multiplication inherited from $V$. The quotient comes equipped with a natural linear map

$$\begin{aligned} \pi : V &\longrightarrow V/W \\ \mathbf{v} &\longmapsto [\mathbf{v}] = \mathbf{v} + W, \end{aligned}$$

called the *projection,* which has $\ker \pi = W$.

(a) Suppose $V$ is finite-dimensional, and let $U$ be a subspace complementary to $W$, that is a subspace such that $V = W \oplus U$. Show that the restriction of projection to $U$

$$\pi_U : U \longrightarrow V/W$$

is an isomorphism (hint: you have already done the heavy lifting for this problem last week!)

(b) Now, let $V$ and $V'$ be vector spaces, $T : V \to V'$ a linear map, and $U$ and $U'$ subspaces of $V$ and $V'$ respectively, such that $T(U) \subset U'$ (Note: $T(U)$ is the image of $T$ when restricted to $U$). Finally, let $\pi_V : V \to V/U$ and $\pi_{V'} : V' \to V'/U'$ be the projection maps. Prove that there exists a unique linear map

$$\bar{T} : V/U \longrightarrow V'/U'$$

such that $\bar{T} \circ \pi_V = \pi_{V'} \circ T$. In the special case that $U' = \{\mathbf{0}\}$, this is the map $\bar{T}$ we constructed in class.

(c) Let $C^\infty(\mathbb{R})$ denote the vector space of *infinitely differentiable* functions $f : \mathbb{R} \to \mathbb{R}$ (namely, $f \in C^\infty(\mathbb{R})$ if $f'$ exists and is continuous, $f''$ exists and is continuous, and so on. Examples include many of the standard functions you know—sin, cos, polynomials, exponentials, etc.)

Let $U$ denote the subspace of $C^\infty(\mathbb{R})$ consisting of functions which vanish at 3 and 5

$$U = \{f \in C^\infty(\mathbb{R}) \mid f(3) = f(5) = 0\}$$

(you do not need to prove $U$ is a subspace). Prove that the quotient vector space $C^\infty(\mathbb{R})/U$ is finite-dimensional. What is its dimension? (note that $C^\infty(\mathbb{R})$ is infinite dimensional!)

**2.** Assume that $T \in \mathcal{L}(V)$ (As a reminder, $\mathcal{L}(V)$ is shorthand for $\mathcal{L}(V, V)$, the vector space of linear maps from $V$ to $V$). Let $T^2$ denote the composition $T \circ T$. As usual, we will use the terminology $\ker T$ for the kernel of $T$ and $\operatorname{im} T$ for the image, in contrast to Axler's Null $T$ and range $T$ respectively.

(a) Give an example of a vector space $V$ and a linear operator $T \in \mathcal{L}(V)$, other than $I$ or 0, such that $T^2 = T$.

(b) Prove that if $T^2 = T$, then $V = \ker T \oplus \ker(T - I)$.

(c) Prove that if $V = \ker T + \ker(T - I)$, then $T^2 = T$.

(d) Give an example of a vector space $V$ and a linear operator $T \in \mathcal{L}(V)$ such that $T^2 = -I$.

**3.** Let $U$, $V$, and $W$ be vector spaces, with $V$ and $W$ finite dimensional. Let $T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, U)$.

(a) Prove that $\dim(\operatorname{im} ST) \leq \dim(\operatorname{im} T)$.

(b) Prove that $\dim(\operatorname{im} ST) = \dim(\operatorname{im} T)$ if and only if

$$\operatorname{im} T + \ker S = \operatorname{im} T \oplus \ker S.$$

(c) Prove that $\dim(\ker ST) \leq \dim(\ker S) + \dim(\ker T)$.

**4.** Let $\mathcal{P}_m(\mathbb{R})$ denote the vector space of polynomials with real coefficients with degree at most $m$, and let $T : \mathcal{P}_m(\mathbb{R}) \to \mathcal{P}_m(\mathbb{R})$ be the linear transformation taking any polynomial $p(x)$ to the polynomial

$$(T(p))(x) = (x - 3)p''(x).$$

Above, $p''(x)$ denotes the second derivative $\frac{d^2p}{dx^2}$. Exhibit a matrix for $L$ relative to a suitable basis for $\mathcal{P}_m(\mathbb{R})$, and determine the kernel and image of $T$ (along with their dimensions).