# Math 113 — Homework 3

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**Book problems:**

4. To show that $V = \text{null}(T) \oplus \{au, a \in \mathbb{F}\}$, we need to show that $V = \text{null}(T) + \{au, a \in \mathbb{F}\}$ and that $\text{null}(T) \cap \{au, a \in \mathbb{F}\} = \{0\}$.

   Let $v$ be any element of $V$. Then $v = (v - \frac{T(v)}{T(u)}u) + \frac{T(v)}{T(u)}u$. Here, we have used that $T(u) \neq 0$. But

   $$T(\frac{T(v)}{T(u)}u) = \frac{T(v)}{T(u)}T(u) = T(v),$$

   so $(v - \frac{T(v)}{T(u)}u) \in \text{null}(T)$. We also have that $\frac{T(v)}{T(u)}u \in \{au, a \in \mathbb{F}\}$, so $v \in \text{null}(T) + \{au, a \in \mathbb{F}\}$. Therefore $V = \text{null}(T) + \{au, a \in \mathbb{F}\}$.

   We know that $T(u)$ is not 0, so $T(au) = aT(u)$ is zero if and only if $a = 0$ (Can you show this from the field axioms if required?). Hence the intersection of $\text{null}(T)$ and $\{au, a \in \mathbb{F}\}$ is exactly $\{0\}$.

   Therefore $V = \text{null}(T) \oplus \{au, a \in \mathbb{F}\}$, as required.

9. Consider an arbitrary element $v = (x_1, x_2, x_3, x_4)$ of $\text{null}(T)$. We know that $x_1 = 5x_2$ and that $x_3 = 7x_4$. Therefore

   $$\begin{aligned} v &= (5x_2, x_2, 7x_4, x_4) \\ &= x_2(5,1,0,0) + x_4(0,0,7,1) \end{aligned}$$

   The vectors $(5,1,0,0)$ and $(0,0,7,1)$ are linearly independent, as neither is a multiple of the other, so they are a basis of $\text{null}(T)$. Therefore $\text{null}(T)$ is two-dimensional.

   By the rank-nullity theorem, we have that $\dim(\text{im}(T)) = \dim(\mathbb{F}^4) - \dim(\text{null}(T)) = 4 - 2 = 2$. Therefore the image of $T$ is a two-dimensional subspace of $\mathbb{F}^2$.

   Consider a basis of $\text{im}(T)$. This has length two. It is linearly independent in $\mathbb{F}^2$, so it's a basis of $\mathbb{F}^2$. (Any linearly independent set of size $n$ in a vector space of dimension $n$ is a basis). Therefore any element of $\mathbb{F}^2$ is a linear combination of elements of $\text{im}(T)$, so is in $\text{im}(T)$. Thus $\text{im}(T)$ is all of $\mathbb{F}^2$, so $T$ is surjective.

15. Assume that $T$ is surjective. The rank-nullity theorem tells us that $\text{im}(T)$ is finite-dimensional, because the domain $V$ is finite-dimensional. The map $T$ is surjective, so $\text{im}(T) = W$, so $W$ is finite-dimensional. Let $w_1, w_2, \ldots, w_n$ be a basis for $W$. The map $T$ is surjective, so for each $w_i$, there is a $v_i \in V$ with $T(v_i) = w_i$.

    Define a linear map $S$ from $W$ to $V$ by $S(w_i) = v_i$ for each $w_i$ (Recall that we may define a linear map uniquely by giving its value on each element of a basis). The composite map $TS$ satisfies $TS(w_i) = T(v_i) = w_i$ for each $w_i$, so $TS \in \mathcal{L}(W)$ is the identity on each $w_i$, so is the identity map. Therefore if $T$ is surjective then there is some $S$ with $TS = 1_W$.

    Now, assume that there is some $S \in \mathcal{L}(W, V)$ with $TS = 1_W$. Then for any $w \in W$, we have that $T(S(w)) = w$, so $w$ is in the image of $T$, so $T$ is surjective.

    Therefore $T$ is surjective if and only if there is some $S \in \mathcal{L}(W, V)$ with $TS = 1_W$, as required.

22. A map is invertible if and only if it is both injective and surjective. Recall also that a linear map from $V$ to $V$ is injective if and only if it is surjective, for any finite-dimensional space $V$.

    Assume that $TS$ is not injective. Then there are $v$ and $w$ in $V$ with $T(S(v)) = T(S(w))$ and $v \neq w$. Therefore either $S(v) = S(w)$, so $S$ is not injective, or $S(v) \neq S(w)$, in which case $T$ is not injective.

    Assume that $TS$ is not surjective. Then there is $v$ in $V$ with $v \notin \mathrm{im}(TS)$. If $T$ is surjective, then there is $w \in V$ with $T(w) = v$. If $w$ was in the image of $S$, then $v$ would be in the image of $TS$, a contradiction. Therefore $w$ is not in the image of $S$, so $S$ is not surjective. Therefore either $T$ or $S$ is not surjective.

    (Only one of the above two arguments is needed. The other can be done by quoting the result that $TS$ is injective if and only if it is surjective, because $V$ is finite-dimensional).

    Assume that $S$ is not injective. Then there are $v \neq w$ in $V$ with $S(v) = S(w)$. Then $T(S(v)) = T(S(w))$, so $TS$ is not injective.

    Assume that $S$ is not surjective. Then $S$ is not injective, so $TS$ is not injective, by the result of the previous paragraph.

    Assume that $T$ is not surjective. Then there is some $v \in V$ which is not in the image of $T$. But the image of $TS$ is contained in the image of $T$, so $v$ is not in the image of $TS$, so $TS$ is not surjective.

    Assume that $T$ is not injective. Then $T$ is not surjective, so $TS$ is not surjective, by the previous paragraph.

    We have shown that if $TS$ fails to be invertible then one of $T$ or $S$ is not invertible, and that if either of $T$ or $S$ fails to be invertible, then $TS$ is not invertible. Therefore $TS$ is invertible if and only if both $S$ and $T$ are invertible.

23. Assume that $ST = I$. The identity map $I$ is invertible with inverse $I$, so by the result of the previous question, both $S$ and $T$ are invertible. Thus there exists a map $T^{-1}$ with $T^{-1}T = TT^{-1} = I$. We have that

    $$S = SI = S(TT^{-1}) = (ST)T^{-1} = T^{-1}.$$

    Therefore $S = T^{-1}$, so $TS = I$. Repeating this proof with the roles of $T$ and $S$ interchanged proves the converse result. Therefore $ST = I$ if and only if $TS = I$.

24. Choose a basis of $V$, which is finite dimensional. Let the dimension of $V$ be $n$. We will work with $n \times n$ matrices $A$ and $B$ representing the linear transformations $T$ and $S$. It suffices to show that an $n \times n$ matrix $A$ commutes with all $n \times n$ matrices if and only if $A$ is a scalar multiple of the identity matrix.

    Let $A = kI$, for some scalar $k$. Then for any $n \times n$ matrix $B$, we have that $AB = kB = BA$.

    Now, assume that $A$ is an $n \times n$ matrix which commutes with every $n \times n$ matrix. For each $i$ with $1 \leq i \leq n$, let $B_i$ be a matrix whose $(i, i) - -$entry is 1 and all of whose other entries are zero. Then $AB_i = B_iA$ for each $i$. But $AB_i$ is a matrix whose $i$th column is the same as that of $A$, and with all other entries zero, while $B_iA$ is a matrix whose $i$th row is the same as that of $A$, with all other entries zero. For these two matrices to be equal, each of their entries are zero, so the only nonzero entry in the $i$th row or column of $A$ is the $(i, i)$–entry. Doing this for each $i$, we get that the only nonzero entries of $A$ are on the diagonal. Let the entries on the diagonal of $A$ be $k_1, k_2, \ldots, k_n$.

    Now, for each $j$ with $1 < j \leq n$, let $C_j$ be the matrix with the $(1, j)$– and $(j, 1)$–entries equal to 1, and all other entries zero. We have that $AC_j = C_jA$ for each $j$. Calculating these matrix products, we get that the $(1, j)$–entry of $AC_j$ is $k_1$, while the $(1, j)$–entry of $C_jA$ is $k_j$. For these to be equal, $k_1 = k_j$. This is true for each $j$, so all of the diagonal entries of $A$ are equal. Hence $A$ is $k_1$ times the identity matrix, as required.

    (You should do the matrix multiplications which are used in this question, in order to understand what multiplying by $B_i$ or $C_j$ on the right or on the left does to a matrix.)

**Other problems:**

1. (a) The map $\pi_U$ is a linear map from $U$ to $V/W$. The kernel of $\pi$ is $W$, so the kernel of $\pi_U$ is $W \cap U$. We know that $V = W \oplus U$, so $W \cap U = \{0\}$. Therefore the kernel of $\pi_U$ is $\{0\}$, which has dimension zero. Therefore $\pi_U$ is injective.

Let the dimensions of $U$ and $W$ be $m$ and $n$ respectively. We have that $V = W \oplus U$, so $\dim(V) = m+n$.

By the rank-nullity theorem,

$$\begin{aligned}\dim(\mathrm{im}(\pi_U)) &= \dim(U) - \dim(\ker(\pi_U)) \\ &= m - 0 \\ &= m\end{aligned}$$

But the image of $\pi_U$ is inside $V/W$, which we proved in last week's homework to have dimension $\dim(V) - \dim(W) = (m+n) - n = m$. Therefore the image of $\pi_U$ is all of $V/W$, by the same argument as in Axler question 9, above. Hence $\pi_U$ is surjective.

We have shown that the linear map $\pi_U$ is both injective and surjective, so it is an isomorphism.

(b) We will construct a map $\bar{T}$ with the required properties, and then show that it is unique.

Consider an arbitrary element $v+U$ of $V/U$. Define the map $\bar{T} : V/U \longrightarrow V'/U'$ by $\bar{T}(v+U) = T(v)+U'$. (Make sure that you understand why the right hand side of this expression is an element of $V'/U'$).

We need to check that this map is well-defined. That is, that it does not depend on which representative $v$ of the equivalence class $v+U$ was chosen. To show this, let $v'$ be another element of the equivalence class $v+U$. That is, $(v'-v) \in U$. We need to show that regardless of whether we consider the equivalence class as $v+U$ or as $v'+U$, applying $\bar{T}$ gives the same result. We calculate that

$$\begin{aligned}\bar{T}(v'+U) &= T(v') + U' \\ &= T(v + (v'-v)) + U' \\ &= T(v) + T(u) + U' \quad \text{for some } u \in U. \\ &= T(v) + U' \quad \text{(because } T \text{ maps } U \text{ into } U'\text{)} \\ &= \bar{T}(v+U)\end{aligned}$$

Therefore the map $\bar{T}$ is well-defined. All that remains is to show that $\bar{T} \circ \pi_V = \pi_{V'} \circ T$. Consider any element $v$ of $V$. Then

$$\begin{aligned}\bar{T}(\pi_V(v)) &= \bar{T}(v+U) \\ &= T(v) + U' \\ &= \pi_{V'}(T(v))\end{aligned}$$

Therefore $(\bar{T} \circ \pi_V)(v) = (\pi_{V'} \circ T)(v)$ for any $v \in V$, so the maps $\bar{T} \circ \pi_V$ and $\pi_{V'} \circ T$ are equal, as required.

We now need to show that $\bar{T}$ is unique. Assume that there were two maps, $\bar{T}_1$ and $\bar{T}_2$ from $V/U$ to $V'/U'$ with $\bar{T}_i \circ \pi_V = \pi_{V'} \circ T$ for both $i = 1$ and $i = 2$.

Consider any element $v+U$ of $V/U$, $v \in V$. Then for both $i = 1$ and $i = 2$, we have that

$$\begin{aligned}\bar{T}_i(v+U) &= \bar{T}_i(\pi_V(v)) \\ &= \pi_{V'}(T(v))\end{aligned}$$

Therefore $\bar{T}_1(v+U) = \bar{T}_2(v+U)$ for any element $v+U$ of $V/U$. Hence the maps $T_1$ and $T_2$ are equal. Therefore there is at most one map satisfying the given conditions. We have constructed such a map, so there is a unique map with the required properties.

(c) We will show that the space $C^\infty(\mathbb{R})/U$ is has dimension two by exhibiting a basis of length two. Let $f$ be the function $f(x) = \frac{5-x}{2}$ and $g$ be the function $g(x) = \frac{x-3}{2}$. Note that $f$ and $g$ are both in $C^\infty(\mathbb{R})$, and that $f(3) = 1, f(5) = 0, g(3) = 0$ and $g(5) = 1$.

Let $[f]$ and $[g]$ be the equivalence classes in $C^\infty(\mathbb{R})/U$ containing $f$ and $g$, respectively. Consider an arbitrary element $[h]$ of $C^\infty(\mathbb{R})/U$, $h \in C^\infty(\mathbb{R})$. We have that $[h] = [h(3)f + h(5)g]$, as the function $(h - h(3)f - h(5)g)$ is zero at the points $x = 3$ and $x = 5$, so is in $U$. So $[h] = h(3)[f] + h(5)[g]$, showing that $[f]$ and $[g]$ span the space $C^\infty(\mathbb{R})/U$.

Neither of $[f]$ and $[g]$ is a multiple of the other, because they are zero at different points. Hence the set $\{[f], [g]\}$ is linearly independent. We have already shown that this set spans $C^\infty(\mathbb{R})/U$, so it is a basis. The space $C^\infty(\mathbb{R})/U$ has a basis of length two, so it is two-dimensional.

2. (a) Let $V = \mathbb{R}^2 = \{(x, y), x, y \in \mathbb{R}\}$. Define $T(x, y) = (x, 0)$. Then for any $(x, y) \in \mathbb{R}^2$, we have that

$$\begin{aligned} T^2(x,y) &= T(T(x,y)) \\ &= T(x,0) \\ &= (x,0) \\ &= T(x,y) \end{aligned}$$

Therefore $T^2 = T$.

(b) Consider any element $v$ of $V$. Then $v = (v - T(v)) + T(v)$. We calculate that

$$\begin{aligned} T(v - T(v)) &= T(v) - T^2(v) \\ &= T(v) - T(v) \\ &= 0 \end{aligned}$$

and that

$$\begin{aligned} (T - I)(T(v)) &= T(T(v)) - I(T(v)) \\ &= T(v) - T(v) \\ &= 0 \end{aligned}$$

Therefore we have that $(v - T(v)) \in \ker(T)$ and $T(v) \in \ker(T - I)$. We have written $v$ as the sum of an element of $\ker(T)$ and an element of $\ker(T - I)$, so $V = \ker(T) + \ker(T - I)$.

Now, consider any vector $v$ in the intersection of $\ker(T)$ and $\ker(T - I)$. Then $T(v) = 0$ and also $(T - I)(v) = 0$ so $T(v) = v$. Combining these, we get that $v = 0$. Therefore $\ker(T) \cap \ker(T - I) = \{0\}$.

We have shown that $V = \ker(T) + \ker(T - I)$ and that $\ker(T) \cap \ker(T - I) = \{0\}$, so $V = \ker(T) \oplus \ker(T - I)$, as required.

(c) Firstly, note that if $w \in \ker(T - I)$, then $(T - I)(w) = 0$ so $T(w) = w$.

Consider an arbitrary element $v$ of $V$. If $V = \ker(T) + \ker(T - I)$, then we may write $v = u + w$, with $u \in \ker(T)$ and $w \in \ker(T - I)$. Then

$$\begin{aligned} T(v) &= T(u + w) \\ &= T(u) + T(w) \\ &= 0 + w \\ &= w \\ T^2(v) &= T(T(v)) \\ &= T(w) \\ &= w \end{aligned}$$

We have shown that $T^2(v) = T(v)$ for any $v \in V$, so $T^2 = T$.

(d) If $V$ is any vector space over the complex numbers $\mathbb{C}$, then the map $T$ which takes any vector $v$ to $iv$ satisfies $T^2(v) = i^2 v = -v$, so $T^2 = -I$.

A slightly more complex (hah!) example is given by $V = \mathbb{R}^2$, and $T$ the map taking $(x, y)$ to $(y, -x)$. Then $T^2(x, y) = (-x, -y)$, so $T^2 = -I$. It is not a coincidence that this example looks similar to the previous one — any vector space over $\mathbb{C}$ is automatically a vector space over $\mathbb{R}$, and any $\mathbb{C}$–linear map is $\mathbb{R}$–linear.

3. (a) Let $S_{\mathrm{im}(T)}$ be the restriction of $S$ to $\mathrm{im}(T)$. The map $S_{\mathrm{im}(T)}$ is a surjective linear function from the image of $T$ to the image of $ST$. (Check that you can justify each part of this statement). The kernel of $S_{\mathrm{im}(T)}$ is $\ker(S) \cap \mathrm{im}(T)$.

The image of $T$ is finite dimensional, as it's a subspace of the finite-dimensional space $W$, so we may apply the rank-nullity theorem to the map $S_{\mathrm{im}(T)}$, whose range we know to be all of $\mathrm{im}(ST)$, to deduce that

$$\dim(\mathrm{im}(T)) = \dim(\mathrm{im}(ST)) + \dim(\ker(S) \cap \mathrm{im}(T)).$$

Each term in this expression is nonnegative, so we have that $\dim(\mathrm{im}(T)) \geq \dim(\mathrm{im}(ST))$, as required.

(b) We showed in the previous part that

$$\dim(\mathrm{im}(T)) = \dim(\mathrm{im}(ST)) + \dim(\ker(S) \cap \mathrm{im}(T)).$$

Hence $\dim(\mathrm{im}(T)) = \dim(\mathrm{im}(ST))$ if and only if the intersection $\ker(S) \cap \mathrm{im}(T)$ is the zero-dimensional subspace $\{0\}$ of $W$. But $\mathrm{im}(T) + \ker(S)$ is a direct sum if and only if $\ker(S) \cap \mathrm{im}(T) = \{0\}$.

Therefore $\dim(\mathrm{im}(T)) = \dim(\mathrm{im}(ST))$ if and only if $\mathrm{im}(T) + \ker(S) = \mathrm{im}(T) \oplus \ker(S)$, as required.

(c) In the first part of this question, we showed that

$$\dim(\mathrm{im}(T)) = \dim(\mathrm{im}(ST)) + \dim(\ker(S) \cap \mathrm{im}(T)).$$

Therefore

$$\dim(\mathrm{im}(T)) \leq \dim(\mathrm{im}(ST)) + \dim(\ker(S)).$$

Let $n$ be the dimension of $V$. Then

$$-\dim(\mathrm{im}(T)) \geq -\dim(\mathrm{im}(ST)) - \dim(\ker(S))$$

and so

$$(n - \dim(\mathrm{im}(T))) \geq (n - \dim(\mathrm{im}(ST))) - \dim(\ker(S)).$$

Applying the rank-nullity theorem to the maps $T$ and $ST$, we have that

$$\dim(\ker(T)) \geq \dim(\ker(ST)) - \dim(\ker(S)).$$

Rearranging yields

$$\dim(\ker(ST)) \leq \dim(\ker(T)) + \dim(\ker(S)).$$

4. Consider the standard basis $\{1, x, x^2, \ldots, x^m\}$ of $P_m(\mathbb{R})$. We calculate that $T(1) = T(x) = 0$ and that for $k \geq 2$,

$$\begin{aligned} T(x^k) &= (x-3)k(k-1)x^{k-2} \\ &= k(k-1)x^{k-1} - 3k(k-1)x^{k-2} \end{aligned}$$

Therefore the matrix of $T$ with respect to this basis has entries $a_{ij}$ as follows, $1 \leq i, j \leq m+1$.

- $a_{i1} = a_{i2} = 0$ for each $i$.
- For each $i$ with $3 \leq i \leq m+1$, $a_{i-1,i} = k(k-1)$ and $a_{i-2,i} = -3k(k-1)$. Each other $a_{ij}$ is zero.

That is, the first two columns are zero, and each further column has exactly two nonzero entries, one and two places above the diagonal, which differ by a factor of $-3$.

For example, when $m = 4$, the matrix of $T$ is as follows:

$$T = \begin{pmatrix} 0 & 0 & -6 & 0 & 0 \\ 0 & 0 & 2 & -18 & 0 \\ 0 & 0 & 0 & 6 & -36 \\ 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Consider $T$ as the composition of two maps, $A$ and $B$, where $A$ is the map defined by differentiating a polynomial twice, and $B$ is the map defined by multiplication by $(x - 3)$.

Note that the kernel of $A$ is exactly those polynomials of degree at most one, while $B$ is injective. Hence the kernel of $T$ is exactly the kernel of $A$, that is, polynomials of degree at most one. This is a two-dimensional space.

By the rank-nullity theorem, the image of $T$ has dimension $(m + 1) - 2 = (m - 1)$. The image of $A$ is exactly those polynomials of degree at most $m - 2$, and so the set $\{p(x)(x - 3), p(x) \in P_{m-2}(\mathbb{R})\}$ is in the image of $T$. But this is an $(m - 1)$–dimensional subspace of the image of $T$, which is itself only $(m - 1)$–dimensional. Therefore this set is exactly the image of $T$.

We have shown that the kernel of $T$ is the set of polynomials of degree at most 1, and that the image of $T$ is the set of polynomials of degree at most $(m - 1)$ which are multiples of $(x - 3)$.