Math 113 Homework 4

Due Friday, May 3, 2013 by 4 pm

Please remember to write down your name and Stanford ID number, and to staple your solutions. Solutions are due to the Course Assistant, Graham White, in his office, 380-380R (either hand your solutions directly to him or leave the solutions under his door). As usual, please justify all of your solutions and/or answers with carefully written proofs.

Book problems: Solve Axler Chapter 5 problems 3, 4, 7, 12, 13, 18, 20 (pages 94-96).

1. Let $C^\infty(\mathbb{R}, \mathbb{C})$ be the vector space (over $\mathbb{C}$) of complex-valued functions $f : \mathbb{R} \to \mathbb{C}$ that are infinitely differentiable. Let V be the set of functions satisfying the differential equation $f'' = -f$:

$$V = \{f \in C^\infty(\mathbb{R}, \mathbb{C}) | f'' = -f\}.$$

(a) Prove that V is a subspace of $C^\infty(\mathbb{R}, \mathbb{C})$.

(b) In a course on differential equations, you would learn how to prove that that the space of solutions V is *at most* two-dimensional (this related to the fact that the differential equation is *second-order* in f). For this problem, let us simply assume the fact that $\dim V \leq 2$.

Then, prove that the functions $\sin x$, and $\cos x$ both lie in V, and that the list $(\sin x, \cos x)$ forms a basis for V.

(c) Now, consider the linear operator D on $C^\infty(\mathbb{R}, \mathbb{C})$ defined by $D(f) = f'$. Prove that V is an invariant subspace for D.

(d) Finally, consider $D \in \mathcal{L}(V)$ as an operator defined on V (still with the same definition: $D(f) = f'$). Find a basis for V consisting of eigenvectors for D. What are their eigenvalues?

2. *Duals of linear transformations.* Recall that the *dual* of a vector space V, denoted V^*, is the vector space $\mathcal{L}(V, \mathbb{F})$ of linear transformations from V to $\mathbb{F}$ (also called *functionals*). On Homework 2, you studied some first properties of the dual, and proved that if V is finite-dimensional, then V^* is too and $\dim V^* = \dim V$. More explicitly, given a basis $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ of V, one can associated a basis of V^* called the *dual basis* $(\mathbf{v}_1^*, \ldots, \mathbf{v}_n^*)$, where $\mathbf{v}_i^* \in V^* = \mathcal{L}(V, \mathbb{F})$ is the functional determined by its effect on the basis of V

$$\mathbf{v}_i^*(\mathbf{v}_j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise.} \end{cases}$$

Given a linear map $T : V \to W$ between finite-dimensional vector spaces V, W, there is a linear map

$$T^* : W^* \to V^*$$

called the *dual of* T, defined as follows: if $\mathbf{w}^* \in W^*$ is a functional on W, then $T^*\mathbf{w}^* \in V^*$ is the functional on V defined as follows:

$$T^*\mathbf{w}^*(\mathbf{v}) := \mathbf{w}^*(T\mathbf{v})$$

In words, $T^*\mathbf{w}$ is the functional that assigns to a vector $\mathbf{v}$ the value of $\mathbf{w}^*$ applied to $T\mathbf{v}$.

(a) Prove that T^* is a linear map. If $S : W \to U$ is another linear map, then prove that $(ST)^* = T^*S^*$ as maps $U^* \to V^*$.

(b) Let $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and $(\mathbf{w}_1, \ldots, \mathbf{w}_m)$ be bases for V and W respectively, and suppose the matrix of T with respect to these bases has components

$$\mathcal{M}(T, (\mathbf{v}_1, \ldots, \mathbf{v}_n), (\mathbf{w}_1, \ldots, \mathbf{w}_m))_{ij} = a_{ij} \text{ for } 1 \leq i \leq m,\ 1 \leq j \leq n.$$

What are the components of the matrix of T^* with respect to the dual bases of V^* and W^*? (Note: the matrix for T has dimensions $m \times n$, but the matrix for T^* will have dimension $n \times m$!).

(c) Now, let $T : V \to V$, and suppose $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is a basis for which the matrix of T is upper triangular. Find a basis of V^* for which the matrix of T^* is upper triangular. What is the relationship between the eigenvalues of T and T^*?

3. *The dual of the dual.* In what follows, let V be a finite dimensional vector space over $\mathbb{F}$.

(a) For a fixed vector $\mathbf{v} \in V$, let $\text{eval}_{\mathbf{v}} : V^* \to \mathbb{F}$ be the function defined by

$$\text{eval}_{\mathbf{v}}(f) = f(v).$$

(Note that $\text{eval}_{\mathbf{v}}$ is a function from V^* to $\mathbb{F}$, not a function from V to $\mathbb{F}$). In words, $\text{eval}_{\mathbf{v}}$ is the function that associates, to a functional f, its value $f(\mathbf{v})$ at the vector $\mathbf{v}$.

Prove that $\text{eval}_{\mathbf{v}}$ is linear.

(b) By the previous part, for any vector $\mathbf{v} \in V$, we have an element $\text{eval}_v \in \mathcal{L}(V^*, \mathbb{F})$. Thus, define a function

$$E : V \to (V^*)^* = \mathcal{L}(V^*, \mathbb{F})$$

by the prescription

$$E(\mathbf{v}) = \text{eval}_{\mathbf{v}}.$$

Prove that E is a linear transformation.

(c) Prove that E is injective.

(d) Prove that E is surjective.

Together, the previous two parts show that E is an isomorphism between V and $(V^*)^* = \mathcal{L}(V^*, \mathbb{F})$. E is an example of a *canonical isomorphism* (meaning an invertible linear map that can be defined without resort to a basis).

4. *Similar matrices and diagonalization.*

(a) Two $n \times n$ matrices A, B are said to be *similar* if $A = C \cdot B \cdot C^{-1}$ for an invertible $n \times n$ matrix C. Prove that if $T : V \to V$ is a linear transformation, and $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$, $(\mathbf{w}_1, \ldots, \mathbf{w}_n)$ are two bases of V, then $\mathcal{M}(T, (\mathbf{v}_1, \ldots, \mathbf{v}_n))$ and $\mathcal{M}(T, (\mathbf{w}_1, \ldots, \mathbf{w}_n))$ are similar matrices.

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation

$$(x, y) \mapsto (7x - 2y, 4x + y).$$

Find an eigenbasis of $\mathbb{R}^2$ with respect to the linear transformation T. Use this to exhibit the matrix M of T with respect to the standard basis as similar to a diagonal matrix D, i.e., as equal to

$$A^{-1}DA$$

for some diagonal matrix D and some invertible matrix A. In this situation, we say that the matrix M is *diagonalizable*.

Please only use techniques we have developed so far in class (i.e., if you have previously seen how to find eigenvalues using the *determinant*, please do not use this method).