Math 113 — Homework 4

Graham White

May 5, 2013

Book problems:

3. The claim is true. Let U be any subspace of V other than $\{0\}$ and V itself. Then U has a basis $\{u_1, u_2, \ldots, u_m\}$. We have that $m \geq 1$, because $U \neq \{0\}$. Extend this to a basis of V,

$$\{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_{n-m}\}.$$

We have that $(n - m) \geq 1$, because $U \neq V$.

Define a linear operator T by $T(u_1) = v_1$, $T(u_i) = 0$ for each $i \geq 2$, and $T(v_i) = 0$ for each $i \geq 1$. (Recall that there is a unique linear operator defined by a given action on each element of a basis). The operator T does not fix the subspace U, so if U is not equal to $\{0\}$ or to V, then U is not fixed by every linear operator on V.

Therefore if U is fixed by every linear operator on V, then either $U = \{0\}$ or $U = V$, as required.

4. Let λ be any element of $\mathbb{F}$ and let v be any element of $\ker(T - \lambda I)$. It suffices to show that $S(v) \in \ker(T - \lambda I)$.

$$\begin{aligned}
(T - \lambda I)(S(v)) &= T(S(v)) - \lambda(S(v)) \\
&= S(T(v)) - \lambda(S(v)) && (TS = ST) \\
&= S(T(v)) - S(\lambda I(v)) && (S \text{ is linear}) \\
&= S(T(v) - \lambda I(v)) && (S \text{ is linear} \\
&= S((T - \lambda I)(v)) \\
&= S(0) && v \in \ker(T - \lambda I) \\
&= 0 && (S \text{ is linear}
\end{aligned}$$

Therefore $S(v) \in \ker(T - \lambda I)$, so the subspace $\ker(T - \lambda I)$ is invariant under S, as required.

7. Let $e_1, e_2, \ldots, e_n$ be the standard basis vectors. We will show that the set

$$\{e_2 - e_1, e_3 - e_2, \ldots, e_n - e_{n-1}, e_1 + e_2 + \ldots + e_n\}$$

is linearly independent.

Assume that

$$a_1(e_2 - e_1) + a_2(e_3 - e_2) + \ldots + a_{n-1}(e_n - e_{n-1}) + a_n(e_1 + e_2 + \ldots + e_n) = 0.$$

Rearranging, we get that

$$(a_n - a_1)e_1 + (a_n + a_1 - a_2)e_2 + (a_n + a_2 - a_3)e_3 + \ldots + (a_n + a_{n-2} - a_{n-1})e_{n-1} + (a_n + a_{n-1})e_n. \quad (1)$$

But the set $\{e_1, e_2, \ldots, e_n\}$ is linearly independent, so each coefficient of Equation 1 is equal to 0. Hence we have that $a_1 = a_n$ and that $a_2 = 2a_n$, and that $a_3 = 3a_n$, $\ldots$, and that $a_{n-1} = (n-1)a_n$. But then we have that $a_n = -a_{n-1}$, so $a_n = 0$, which gives us that each a_i is zero. Hence the set

$$\{e_2 - e_1, e_3 - e_2, \ldots, e_n - e_{n-1}, e_1 + e_2 + \ldots + e_n\}$$

is linearly independent, as claimed.

A linearly independent set with n elements in a vector space of dimension n is a basis. Now the $(n-1)$ elements of the form $(e_k - e_{k-1})$ are eigenvectors of T with eigenvalue 0. (You should verify this by actually applying T to these vectors). The final element of our basis, $(e_1 + e_2 + \ldots + e_n)$, is an eigenvector of T with eigenvalue n. (Similarly, you should verify this).

We have a basis of $\mathbb{F}^n$ comprised of eigenvectors of T. With respect to this basis, the matrix of T is diagonal, so the only eigenvalues of T are 0 and n (by Proposition 5.18 of Axler, for example). The 0–eigenspace of T is not all of $\mathbb{F}^n$, because n is an eigenvalue of T. Therefore the 0–eigenspace of T is at most $(n-1)$–dimensional. But we know that the 0–eigenspace of T contains the span of the $(n-1)$ linearly independent vectors $\{e_2 - e_1, e_3 - e_2, \ldots, e_n - e_{n-1}\}$. Therefore this is exactly the 0–eigenspace of T. (This can also be described as the set of vectors whose coordinates sum to zero).

The n–eigenspace of T has dimension at most $n - \dim(\ker(T)) = 1$, because the intersection of two distinct eigenspaces is $\{0\}$. Therefore the n–eigenspace of T is exactly the one-dimensional space spanned by $(e_1 + e_2 + \ldots + e_n)$.

(This question could also be done by a direct calculation, but some of the theory used here is instructive.)

12. (The following argument does not assume that V is finite-dimensional). Consider any two elements v and w of V. Both v and w are eigenvectors of T. Let the corresponding eigenvalues be a and b.

    If w is a multiple of v, then $a = b$, as T is linear.

    We now consider the case when v and w are linearly independent. The vector $(v + w)$ is an eigenvector of T. Therefore $T(v + w) = c(v + w)$ for some scalar c.

    We have that

$$T(v + w) = c(v + w)$$
$$av + bw = cv + cw$$

    Therefore $a = c$ and $b = c$, because v and w are linearly independent. Therefore v and w are eigenvectors of T with the same eigenvalues. But v and w were arbitrary elements of V, so any element of V is an eigenvector of T with the same eigenvalue. Let this eigenvalue be λ. Then $T(v) = (\lambda I)(v)$ for each vector $v \in V$, so $T = \lambda I$, as required.

13. Let $\dim(V) = n$. It suffices to show that if T is not a scalar multiple of the identity, then there is some subspace U of V with $\dim(U) = (n-1)$ and $T(U) \nsubseteq U$.

    Assume that T is not a scalar multiple of the identity. Then from the previous question, there is some vector $v \in V$ which is not an eigenvector of T. Then the vectors v and $T(v)$ are linearly independent. Extending this set $\{v, T(v)\}$ to a basis of V, we have a basis of V,

$$\{v, T(v), v_1, v_2, \ldots, v_{n-2}\}.$$

    Let U be the subspace spanned by each of these basis elements other than $T(v)$, that is, $U = \mathrm{Span}(v, v_1, v_2, \ldots, v_{n-2})$. Then $\dim(U) = (n-1)$ and $T(U) \ni T(v) \notin U$, so $T(U) \nsubseteq U$. Therefore U is subspace with the required properties, so the result is proven.

18. Let V be the vector space $\mathbb{R}^2$, $\{v_1, v_2\}$ be any basis of V, and $T \in \mathcal{L}(V)$ be the linear transformation whose matrix with respect to the basis $\{v_1, v_2\}$ is

$$M(T, \{v_1, v_2\}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

    The matrix of the transformation T^2 is

$$M(T^2, \{v_1, v_2\}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore T^2 is the identity map on V, so T is invertible and its inverse is T. Thus T is an example of the required form.

20. Let $\dim(V) = n$. We are given that T has n distinct eigenvalues, so let these be $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. For each i, $1 \leq i \leq n$, let v_i be a nonzero eigenvector corresponding to the eigenvalue λ_i. We showed in lectures that eigenvectors corresponding to distinct eigenvalues are linearly independent, so the set $\{v_1, v_2, \ldots, v_n\}$ is linearly independent. We also have that $\dim(V) = n$, so this set is a basis of V.

    We are given that eigenvectors of T are also eigenvectors of S, so for each i, $1 \leq i \leq n$, let μ_i be the scalar such that $S(v_i) = \mu_i v_i$. Now, let $v = a_1 v_1 + \ldots + a_n v_n$ be an arbitrary element of V. We have that

$$\begin{aligned} ST(v) &= ST(a_1v_1 + \ldots + a_nv_n) \\ &= S(\lambda_1 a_1 v_1 + \ldots + \lambda_n a_n v_n) \\ &= \mu_1\lambda_1 a_1 v_1 + \ldots + \mu_n \lambda_n a_n v_n \\ &= \lambda_1 \mu_1 a_1 v_1 + \ldots + \lambda_n \mu_n a_n v_n \\ &= T(\mu_1 a_1 v_1 + \ldots + \mu_n a_n v_n) \\ &= TS(a_1 v_1 + \ldots + a_n v_n) \\ &= TS(v) \end{aligned}$$

    We have shown that $TS(v) = ST(v)$ for each $v \in V$, so $TS = ST$.

Other problems:

1. (a) We need to show that V is nonempty, closed under addition, and closed under scalar multiplication. Let $\mathbf{0}$ be the zero function. Then $\mathbf{0} \in \mathbb{C}^\infty(\mathbb{R}, \mathbb{C})$, and $\mathbf{0}'' = \mathbf{0} = -\mathbf{0}$. Therefore $\mathbf{0} \in V$.

    Let f and g be arbitrary elements of V. Then $f'' = -f$ and $g'' = -g$. We have that

$$\begin{aligned} (f+g)'' &= f'' + g'' \\ &= (-f) + (-g) \\ &= -(f+g) \end{aligned}$$

    Therefore $(f+g)'' = -(f+g)$, so $(f+g) \in V$. Hence V is closed under addition.

    Let f be any element of V and a be any element of $\mathbb{C}$. We have that $f'' = -f$, so

$$\begin{aligned} (af)'' &= a(f'') \\ &= a(-f) \\ &= -(af) \end{aligned}$$

    Therefore $(af)'' = -(af)$, so $(af) \in V$. Hence V is closed under scalar multiplication.

    We have shown that V is nonempty, closed under addition, and closed under scalar multiplication, so V is a subspace of $\mathbb{C}^\infty(\mathbb{R}, \mathbb{C})$, as required.

   (b) We know that the derivative of $\sin(x)$ is $\cos(x)$, and that the derivative of $\cos(x)$ is $-\sin(x)$. Therefore the second derivative of $\sin(x)$ is $-\sin(x)$ and the second derivative of $\cos(x)$ is $-\cos(x)$. Further, both $\sin(x)$ and $\cos(x)$ are infinitely differentiable, with derivatives just cycling through positive and negative $\sin(x)$ and $\cos(x)$. Therefore both $\sin(x)$ and $\cos(x)$ are in $\mathbb{C}^\infty(\mathbb{R}, \mathbb{C})$, and then in V.

    Assume that $a\sin(x) + b\cos(x) = \mathbf{0}$. Then

$$\begin{aligned} a\sin(0) + b\cos(0) &= \mathbf{0}(0) \\ a \cdot 0 + b \cdot 1 &= 0 \\ b &= 0 \end{aligned}$$

Likewise,

$$\begin{aligned} a\sin(\frac{\pi}{2}) + b\cos(\frac{\pi}{2}) &= \mathbf{0}(\frac{\pi}{2}) \\ a \cdot 1 + b \cdot 0 &= 0 \\ a &= 0 \end{aligned}$$

We have shown that if $a\sin(x) + b\cos(x) = \mathbf{0}$, then $a = b = 0$. Therefore $\sin(x)$ and $\cos(x)$ are linearly independent. We have assumed that the vector space V has dimension at most 2. But V contains a linearly independent set of size 2, so it has dimension at least 2. Therefore the dimension of V is exactly 2, so the linearly independent set $\{\sin(x), \cos(x)\}$ is a basis of V, as required.

(c) Let $f = a\sin(x) + b\cos(x)$ be any element of V. Then

$$\begin{aligned} D(f) &= a(\sin(x)') + b(\cos(x)') \\ &= a\cos(x) - b\sin(x) \end{aligned}$$

Therefore for any $f \in V$, $D(f) \in V$. Hence V is an invariant subspace for D.

(d) Let f and g be defined as follows:

$$\begin{aligned} f(x) &= \cos(x) + i\sin(x) = e^{ix} \\ g(x) &= \cos(x) - i\sin(x) = e^{-ix} \end{aligned}$$

The functions f and g are linear combinations of $\cos(x)$ and $\sin(x)$, so are in V. Note that $D(f) = if$ and $D(g) = -ig$, so f and g are eigenvectors of D with eigenvalues i and $-i$. Eigenvectors corresponding to different eigenvectors are linearly independent, so f and g are linearly independent. The space V has dimension 2, so the set $\{f, g\}$ is a basis for V consisting of eigenvectors of D.

2. (a) We will first prove that T^* is a linear map from W^* to V^*. Let f and g be arbitrary elements of W^* and a be any scalar. Then for any element $v \in V$, we have that

$$\begin{aligned} (T^*(f+g))(v) &= (f+g)(T(v)) \\ &= f(T(v)) + g(T(v)) \\ &= (T^*(f))(v) + (T^*(g))(v) \\ &= (T^*(f) + T^*(g))(v) \end{aligned}$$

We have shown that $(T^*(f+g))(v) = (T^*(f) + T^*(g))(v)$ for each $v \in V$. Therefore the functions $T^*f + g$ and $T^*(f) + T^*(g)$ are equal.

Likewise, for any element $v \in V$, we have that

$$\begin{aligned} (T^*(af))(v) &= (af)(T(v)) \\ &= a(f(T(v))) \\ &= a((T^*(f))(v)) \\ &= (a(T^*(f)))(v) \end{aligned}$$

We have shown that $(T^*(af))(v) = (aT^*(f))(v)$ for each $v \in V$. Therefore the functions T^*af and $a(T^*(f))$ are equal. Hence T^* is linear, as required.

Now, consider any $T : V \longrightarrow W$ and $S : W \longrightarrow U$. Then T^* is a map from W^* to V^* and S^* is a map from U^* to W^*, so the composition T^*S^* is a map from U^* to V^*. We also have that ST is a map from V to U, so $(ST)^*$ is a map from U^* to V^*. Therefore T^*S^* and $(ST)^*$ are both maps from U^* to V^*, so to prove that they are equal, we just need to check that they agree on each element of U^*.

Let f be any element of U^*. We need to show that $T^*S^*(f) = (ST)^*(f)$. Both of these are elements of V^*, so to show that they are equal, we need to show that they agree on any element v of V. We have that for any $v \in V$,

$$\begin{aligned}(T^*S^*(f))(v) &= (S^*(f))(T(v)) \\ &= f(S(T(v))) \\ &= f(ST(v)) \\ &= ((ST)^*(f))(v)\end{aligned}$$

Hence $T^*S^*(f) = (ST)^*(f)$ for each element $f \in U^*$, so $T^*S^* = (ST)^*$ as required. (Make sure that you're comfortable with this calculation, as there's quite a bit going on. Can you identify which set each term is in?)

(b) Let the dual bases for V^* and W^* be $\{v_1^*, v_2^*, \ldots, v_n^*\}$ and $\{w_1^*, w_2^*, \ldots, w_m^*\}$. The matrix of T^* with respect to these bases is defined as follows. We write $T^*(w_i^*)$ as a linear combination of the $\{v_j^*\}$, and the coefficients form the ith column of the matrix.

Each $T^*(w_i^*)$ is an element of V^*, that is, a function from V to $\mathbb{F}$, so is determined by its action on an arbitrary element v of V. Let the matrix of T with respect to the bases $\{v_i\}$ and $\{w_j\}$ have (i,j)–entry a_{ij}. Let $v = b_1v_1 + \ldots + b_nv_n$ be any element of V. Then

$$\begin{aligned}(T^*(w_i^*))(v) &= w_i^*(Tv) \\ &= w_i^*(b_1T(v_1) + \ldots + b_nT(v_n)) \\ &= b_1w_i^*(T(v_1)) + \ldots + b_nw_i^*(T(v_n)) \\ &= b_1w_i^*(a_{11}w_1 + \ldots + a_{m1}w_m) + \ldots + b_nw_i^*(a_{1n}w_1 + \ldots + a_{mn}w_m)) \\ &= b_1a_{i1} + \ldots + b_na_{in} \\ &= a_{i1}v_1^*(v) + \ldots + a_{in}v_n^*(v) \\ &= (a_{i1}v_1^* + \ldots + a_{in}v_n^*)(v)\end{aligned}$$

We have shown that

$$(T^*(w_i^*))(v) = (a_{i1}v_1^* + \ldots + a_{in}v_n^*)(v)$$

for each $v \in V$, so

$$T^*(w_i^*) = a_{i1}v_1^* + \ldots + a_{in}v_n^*.$$

Therefore the ith column of the matrix of T^* with respect to the bases $\{w_j^*\}$ and $\{v_i^*\}$ is $(a_{i1}, \ldots, a_{in})$. But this is the ith row of the matrix of T with respect to the bases $\{v_i\}$ and $\{w_j\}$. Therefore, with respect to these bases, the matrix of T^* is the transpose of the matrix of T. That is, the (i,j)–entry of the matrix of $T*$ is equal to the (j,i)–entry of the matrix of T.

(c) If the matrix of T is upper triangular with respect to the basis $\{v_1, \ldots, v_n\}$, then by the result of the previous part, the matrix of $T*$ with respect to the basis $\{v_1^*, \ldots, v_n^*\}$ is the transpose of this upper triangular matrix, so is lower triangular.

Therefore for each i, the only nonzero entries in the ith column of the matrix of T^* are those in the ith row and below, so

$$T^*(v_i^*) \in \text{Span}(v_i^*, v_{i+1}^*, \ldots, v_n^*).$$

By Proposition 5.12 of Axler, the set $\{v_n^*, v_{n-1}^*, \ldots, v_2^*, v_1^*\}$ is a basis with respect to which the matrix of T^* is upper triangular. Indeed, the (i,j)–entry of this matrix is equal to the $(n+1-i, n+1-j)$–entry of the matrix of T^* with respect to the basis $\{v_1^*, v_2^*, \ldots, v_n^*\}$ (Could you prove this, if required?). Therefore the diagonal entries are just reordered.

The eigenvalues of T are the diagonal entries of the matrix for T, by Proposition 5.18 of Axler. These diagonal entries are preserved under transposition and then reordered under the change of basis, so our matrix for T^* has the same diagonal entries as the matrix for T, just in a different order. Using Proposition 5.18 of Axler again, the eigenvalues of T^* are the same as the eigenvalues of T.

3. (a) Let v be any fixed element of V. To show that $\mathrm{eval_v}$ is linear, let f and g be arbitrary element of V^* and let a be any scalar. Then we have that

$$\begin{aligned}\mathrm{eval_v}(f+g) &= (f+g)(v)\\ &= f(v)+g(v)\\ &= \mathrm{eval_v}(f)+\mathrm{eval_v}(g)\end{aligned}$$

Likewise,

$$\begin{aligned}\mathrm{eval_v}(af) &= (af)(v)\\ &= a(f(v))\\ &= a\,\mathrm{eval_v}(f)\end{aligned}$$

We have shown that $\mathrm{eval_v}(f+g) = \mathrm{eval_v}(f)+\mathrm{eval_v}(g)$ and that $\mathrm{eval_v}(af) = a\,\mathrm{eval_v}(f)$. Therefore $\mathrm{eval_v}$ is a linear map.

(b) The codomain of E is V^{**}. In order to show that two elements of V^{**} are equal, we just need to check that they are equal on each element of V^*. Let u and v be arbitrary elements of V and let a be any scalar. For any $f \in V^*$, we have that

$$\begin{aligned}(E(u+v))(f) &= \mathrm{eval_{u+v}}(f)\\ &= f(u+v)\\ &= f(u)+f(v) \quad (f \text{ is linear})\\ &= \mathrm{eval_u}(f)+\mathrm{eval_v}(f)\\ &= (E(u))(f)+(E(v))(f)\end{aligned}$$

and that

$$\begin{aligned}(E(av))(f) &= \mathrm{eval_{av}}(f)\\ &= f(av)\\ &= af(v) \quad (f \text{ is linear})\\ &= a\,\mathrm{eval_v}(f)\\ &= (aE(v))(f)\end{aligned}$$

We have shown that for each $f \in V^{**}$,

$$(E(u+v))(f) = (E(u))(f)+(E(v))(f)$$

and

$$(E(av))(f) = (aE(v))(f).$$

Therefore $E(u+v) = E(u)+E(v)$ and $E(av) = aE(v)$, so E is linear.

(c) Let $v \neq 0$ be any nonzero element of V. We will show that $E(v) \neq \mathbf{0}$, so E is injective. (Can you show that this implies injectivity for a linear transformation?)

Extend the set $\{v\}$ to a basis $\{v = v_1, v_2, \ldots, v_n\}$ of V. Consider the dual basis of V^* defined in the previous question. Then $(E(v))(v_1^*) = v_1^*(v) = 1$, so $E(v)$ is not the zero function in V^{**}. Therefore the kernel of E is $\{0\}$, so E is injective.

(d) The linear transformation E has domain V and codomain V^{**}. We showed in Homework set 2 that the dimension of W^* is equal to the dimension of W for any vector space W, so the dimension of V^{**} is equal to the dimension of V^*, which is equal to the dimension of V. Therefore the vector spaces V and V^{**} have the same finite dimension. Let $\dim(V) = n$.

By the rank-nullity theorem, the image of E has dimension n, because E is injective so its kernel is $\{0\}$. Therefore the image of E is a dimension n subspace of V^{**}, which has dimension n, so the image of E is all of V^{**}. (Can you prove this? It's been used in several homework sets so far). Therefore E is surjective.

4. (a) Let I be the identity map from V to V. We have that the composite map ITI is equal to T. Thus

$$\mathcal{M}(T, \{v_i\}) = \mathcal{M}(ITI, \{v_i\})$$

The matrix of a composition of maps is the product of the matrices corresponding to those maps. Therefore

$$\mathcal{M}(T, \{v_i\}) = \mathcal{M}(I, \{v_i\}, \{w_i\})\mathcal{M}(T, \{w_i\})\mathcal{M}(I, \{w_i\}, \{v_i\}). \tag{2}$$

Likewise, we have that $I \cdot I = I$, so $\mathcal{M}(I, \{v_i\}, \{w_i\})\mathcal{M}(I, \{w_i\}, \{v_i\}) = \mathrm{Id}_n$. Therefore the matrix $\mathcal{M}(I, \{v_i\}, \{w_i\})$ is invertible and its inverse is $\mathcal{M}(I, \{w_i\}, \{v_i\})$.

Hence Equation 2 is of the form

$$\mathcal{M}(T, \{v_i\}) = C\mathcal{M}(T, \{w_i\})C^{-1}$$

for some $n \times n$ matrix C.

Therefore the matrices $\mathcal{M}(T, \{v_i\})$ and $\mathcal{M}(T, \{w_i\})$ are similar, as required.

(b) To find the eigenvalues and eigenvectors of T, we solve the equation $Tv = \lambda v$. Let $v = \begin{pmatrix} x \\ y \end{pmatrix}$. If $Tv = \lambda v$ then we have the following:

$$\begin{aligned} T\begin{pmatrix} x \\ y \end{pmatrix} &= \lambda \begin{pmatrix} x \\ y \end{pmatrix} \\ \begin{pmatrix} 7x - 2y \\ 4x + y \end{pmatrix} &= \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} \end{aligned}$$

Therefore we have that

$$\begin{aligned} 7x - 2y &= \lambda x \\ 4x + y &= \lambda y \end{aligned}$$

So

$$\begin{aligned} y &= \frac{7 - \lambda}{2}x \\ 4x + \frac{7 - \lambda}{2}x &= \lambda\frac{7 - \lambda}{2}x \\ (\lambda^2 - 8\lambda + 15)x &= 0 \\ (\lambda - 3)(\lambda - 5)x &= 0 \end{aligned}$$

Now, we check that $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \neq \lambda\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is not an eigenvector of T and hence we may assume that $x \neq 0$. Therefore the eigenvalues of T are $\lambda = 3$ and $\lambda = 5$. We have that

$$y = \frac{7 - \lambda}{2}x,$$

so the eigenvectors corresponding to $\lambda = 3$ are $\begin{pmatrix} x \\ 2x \end{pmatrix}$ and those corresponding to $\lambda = 5$ are $\begin{pmatrix} x \\ x \end{pmatrix}$.

Let $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be the standard basis for $\mathbb{R}^2$. We have showed that $w_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $w_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are eigenvectors of T, and they are linearly independent therefore they are a basis of $\mathbb{R}^2$. Hence $\{w_1, w_2\}$ is an eigenbasis of $\mathbb{R}^2$ for T. We have from the previous part that

$$\mathcal{M}(T, \{v_i\}) = \mathcal{M}(I, \{w_i\}, \{v_i\})^{-1}\mathcal{M}(T, \{w_i\})\mathcal{M}(I, \{w_i\}, \{v_i\}).$$

By definition,

$$\mathcal{M}(T, \{v_i\}) = \begin{pmatrix} 7 & -2 \\ 4 & 1 \end{pmatrix}.$$

Note that $w_1 = v_1 + 2v_2$ and that $w_2 = v_1 + v_2$. Therefore

$$\mathcal{M}(I, \{w_i\}, \{v_i\}) = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

We know that $\{w_1, w_2\}$ is an eigenbasis for T, so

$$\mathcal{M}(T, \{w_i\}) = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}.$$

Putting these all together, we have that

$$\begin{pmatrix} 7 & -2 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

This equation shows that the matrix of T with respect to the standard basis is similar to a diagonal matrix.