Math 113 Homework 5

Due Friday, May 10, 2013 by 4 pm

Please remember to write down your name and Stanford ID number, and to staple your solutions. Solutions are due to the Course Assistant, Graham White, in his office, 380-380R (either hand your solutions directly to him or leave the solutions under his door). As usual, please justify all of your solutions and/or answers with carefully written proofs.

Book problems: Solve Axler Chapter 8 problems 3, 6, 8, 14, 15, 20 (pages 188-191).

1. *Abstract operations on vector spaces I: direct sums.* Given a pair of vector spaces V and W, we can form their *formal direct sum*

$$V \oplus W$$

as follows: $V \oplus W$ as a set is the set of pairs of vectors $(\mathbf{v}, \mathbf{w})$, where $\mathbf{v} \in V$, and $\mathbf{w} \in W$. Such a pair is often denoted $\mathbf{v} \oplus \mathbf{w}$, where we are using $\oplus$ here to note that this sum is "formal" (i.e. V and W are not a priori subspaces of the same vector space). Addition is defined by

$$\mathbf{v} \oplus \mathbf{w} + \mathbf{v}' \oplus \mathbf{w}' = (\mathbf{v} + \mathbf{v}') \oplus (\mathbf{w} + \mathbf{w}')$$

and scalar multiplication is

$$c \cdot (\mathbf{v} \oplus \mathbf{w}) = c\mathbf{v} \oplus c\mathbf{w}.$$

Note: it may be helpful to denote this *formal direct sum* with different notation than the direct sum of subspaces, to avoid confusion. You can feel free to refer to the direct sum here as an operation with a different symbol, e.g. $V \underline{\oplus} W$. (where we have underlined the symbol $\oplus$).

(a) Prove that if V and W are both finite dimensional, then $\dim(V \oplus W) = \dim V + \dim W$.

(b) Now let's suppose V and W are both subspaces of a single subspace U. Form the formal direct sum $V \oplus W$ (note: this is not necessarily a subspace of U!). There is a natural linear map

$$T : V \oplus W \longrightarrow U$$

sending a formal sum $(\mathbf{v}, \mathbf{w})$ to $\mathbf{v} + \mathbf{w}$, where the latter expression means to take the sum of $\mathbf{v}$ and $\mathbf{w}$ in the vector space U. Prove this map is linear and determine its kernel and image.

(c) Now, let V, W, and U be any vector spaces (we are no longer requiring W and V to be subspaces of U). Show that there is *canonical* isomorphism of vector spaces

$$\mathcal{L}(V \oplus W, U) \cong \mathcal{L}(V, U) \oplus \mathcal{L}(W, U).$$

Remark: A linear map $T : V \to W$ is said to be *canonical* if it can be defined without using a basis. Canonical maps are important precisely because there are very few examples of them: for most maps from V to W, we resort to defining a linear operator by saying what it does to a basis of V in terms of a basis of W. **Examples**: for maps from V to V, the identity and 0 map are canonical maps. Another example is the projection

$$\begin{aligned} \pi_W : V &\to V/W \\ \mathbf{v} &\mapsto \mathbf{v} + W \end{aligned}$$

for $W \subset V$ some subspace. As demonstrated, we did not need to decompose $\mathbf{v}$ into a linear combination of basis elements to define the above map.

Finally, a *canonical isomorphism* is a canonical linear map that is also an isomorphism (namely, it is invertible, or equivalently, it is injective and surjective).

(d) Verify that the operation of direct sum has an additive identity and is associative. (However, there is no additive inverse!)

Important clarification: Usually, an additive identity for an operation + means that there exists an element 0 with $a + 0 = 0 + a = a$. Here, when we talk about direct sums of vector spaces, the symbol = means *canonically isomorphic*. That is, to verify the existence of an additive identity, you should find a vector space V_0 such that $V \oplus V_0$ and $V_0 \oplus V$ are both *canonically isomorphic* to V, for all V.