Math 113 — Homework 5

Graham White

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Book problems:

3. For the sake of a contradiction, assume that the set $\{v, Tv, T^2v, \ldots, T^{m-1}v\}$ is linearly dependent. Then for some integer k with $0 \le k \le m-1$ there is an equation

$$a_k T^k v = a_{k+1} T^{k+1} v + \ldots + a_{m-1} T^{m-1} v \tag{1}$$

with $a_k \neq 0$ (Could you justify each part of this statement? Why can't we assume that $k = 0$? How do we know that $a_k \neq 0$?).

Applying the linear operator T^{m-k-1} to Equation 1, we get that

$$a_k T^{m-1} v = a_{k+1} T^m v + \ldots + a_{m-1} T^{2m-2-k} v. \tag{2}$$

For any $n \ge m$ we have that $T^n v = T^{n-m} T^m v = T^{n-m} 0 = 0$, so each term on the right of Equation 2 is zero. Therefore we have that

$$a_k T^{m-1} v = 0.$$

But $a_k \neq 0$ and $T^{m-1} v \neq 0$, a contradiction. Therefore our original assumption was incorrect, so the set $\{v, Tv, T^2v, \ldots, T^{m-1}v\}$ is linearly independent, as required.

6. Assume that λ is an eigenvalue of N. Let v be a nonzero λ–eigenvector of N, that is, $Tv = \lambda v$. The operator N is nilpotent, so there is a positive integer n with $N^n = 0$. Then we have that $N^n v = 0$. But v is a λ–eigenvector of T, so $N^n(v) = \lambda^n v$. Combining these, we have that $\lambda^n v = 0$.

The vector v is nonzero, so $\lambda^n = 0$. But in a field, $\lambda^n = 0$ implies that $\lambda = 0$. (Can you show this from the field axioms?). Hence any eigenvalue of N is zero.

8. Let $n = \dim(V)$. By Proposition 5 of Axler, if for some integer k with $0 \le k \le n-1$ we were to have that $\ker(T^k) = \ker(T^{k+1})$, then we would have that $\ker(T^{m-1}) = \ker(T^m)$, a contradiction. Therefore for each k with $0 \le k \le n-1$, we have $\ker(T^k) \subsetneq \ker(T^{k+1})$.

Therefore we have the following ascending chain of strict inclusions

$$\{0\} \subsetneq \ker(T) \subsetneq \ker(T^2) \subsetneq \cdots \subsetneq \ker(T^n) \tag{3}$$

If W_1 and W_2 are subspaces of V with $W_1 \subsetneq W_2$, then $\dim(W_2) \ge \dim(W_1) + 1$ (This lemma has been used in many homework sets. Make sure you know how to prove it!). Therefore the dimensions of the $n+1$ subspaces in Equation 3 are a sequence of strictly increasing integers between 0 and n. But there are only $n+1$ integers in this range, so this sequence must be exactly $\{0, 1, 2, \ldots, n\}$. Therefore $\dim(\ker(T^k)) = k$ for each k with $0 \le k \le n$.

We have that $\dim(\ker(T^n)) = n$, so $\ker(T^n) = V$. Therefore T^n is the zero operator, so T is nilpotent, by definition.

14. Let $T \in \mathcal{L}(\mathbb{C}^4)$ be defined by $T(x_1, x_2, x_3, x_4) = (7x_1, 7x_2, 8x_3, 8x_4)$. Note that the standard basis vectors e_1 and e_2 are in the 7–eigenspace of T, and that e_3 and e_4 are in the 8–eigenspace of T. Therefore the generalised 7– and 8–eigenspaces of T are each at least two-dimensional. Hence the multiplicities of the eigenvalues 7 and 8 are both at least two.

The sum of the multiplicities of all eigenvalues of T is equal to the dimension of $\mathbb{C}^4$, which is 4. Therefore the multiplicities of the eigenvalues 7 and 8 are both exactly two, and T has no other eigenvalues. Thus the characteristic polynomial of T is $(x-7)^2(x-8)^2$, by definition.

15. Let the multiplicities of the eigenvalues 5 and 6 of T be a and b respectively. Then the characteristic polynomial of T is $(x-5)^a(x-6)^b$. By the Cayley-Hamilton theorem, we have that

$$(T-5)^a(T-6)^b = 0. \tag{4}$$

The sum of the multiplicities of the eigenvalues of T is equal to $\dim(V) = n$, so $a+b=n$. In addition, each of a and b are at least 1. Therefore $1 \leq a, b \leq n-1$. Applying $(T-5)^{n-1-a}(T-6)^{n-1-b}$ to Equation 4, we get that $(T-5)^{n-1}(T-6)^{n-1} = 0$, because powers of T commute with one another.

20. Let the characteristic polynomial of T be

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0.$$

We know that T is invertible, so it does not have 0 as an eigenvalue. Therefore x is not a factor of the characteristic polynomial of T, so $a_0 \neq 0$. By the Cayley-Hamilton theorem,

$$a_n T^n + a_{n-1}T^{n-1} + \ldots + a_1 T + a_0 I = 0.$$

Working with this equation,

$$-a_0 I = a_n T^n + a_{n-1}T^{n-1} + \ldots + a_1 T.$$

Dividing by $-a_0$ and applying T^{-1}, we get that

$$T^{-1} = a_n T^{n-1} + a_{n-1}T^{n-2} + \ldots + a_2 T + a_1.$$

This is an equation of the required form.

Other problems:

1. (a) It is easily verified that $V \oplus W$ is a vector space, with identity element $0 = (0_V, 0_W)$.

If V and W are finite dimensional, then we may consider bases $\{v_1, \ldots, v_m\}$ of V and $\{w_1, \ldots, w_n\}$ of W. We will show that the set

$$\{(v_1, 0), \ldots, (v_m, 0), (0, w_1), \ldots, (0, w_n)\} \tag{5}$$

is a basis of the vector space $V \oplus W$.

Consider an arbitrary element (v, w) of $V \oplus W$. By definition, v and w are elements of V and W respectively, so we may write $v = a_1 v_1 + \ldots + a_m v_m$ and $w = b_1 w_1 + \ldots + b_n w_n$. Then we have that

$$(v, w) = a_1(v_1, 0) + \ldots + a_m(v_m, 0) + b_1(0, w_1) + \ldots + b_n(0, w_n).$$

Therefore the set in Equation 5 spans the space $V \oplus W$.

Assume that for some scalars $\{a_i\}$ and $\{b_j\}$, we have a relation

$$0 = a_1(v_1, 0) + \ldots + a_m(v_m, 0) + b_1(0, w_1) + \ldots + b_n(0, w_n).$$

Then

$$(0_V, 0_W) = (a_1 v_1 + \ldots + a_m v_m, b_1 w_1 + \ldots + b_n w_n). \tag{6}$$

By the definition of $V \oplus W$, the corresponding components of Equation 6 are equal. Therefore

$$0_V = a_1 v_1 + \ldots + a_m v_m$$
$$0_W = b_1 w_1 + \ldots + b_n w_n$$

But the sets $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$ are bases of V and W. Therefore they are linearly independent, so we have that each a_i and each b_j are zero.

Therefore the set in Equation 5 is linearly independent, so it is a basis of $V \oplus W$.

We have shown that $V \oplus W$ has a basis of size $m + n$, so

$$\dim(V \oplus W) = \dim(V) + \dim(W).$$

(b) To show that the map T is linear, let (v_1, w_1) and (v_2, w_2) be arbitrary elements of $V \oplus W$ and let a be any scalar. We have that

$$\begin{aligned} T((v_1, w_1) + (v_2, w_2)) &= T((v_1 + v_2, w_1 + w_2)) \\ &= v_1 + v_2 + w_1 + w_2 \\ &= v_1 + w_1 + v_2 + w_2 \\ &= T((v_1, w_1)) + T((v_2, w_2)) \end{aligned}$$

and that

$$\begin{aligned} T(a(v_1, w_1)) &= T((av_1, aw_1)) \\ &= av_1 + aw_1 \\ &= a(v_1 + w_1) \\ &= aT((v_1, w_1)) \end{aligned}$$

We have shown that

$$T((v_1, w_1) + (v_2, w_2)) = T((v_1, w_1)) + T((v_2, w_2))$$

and that

$$T(a(v_1, w_1)) = aT((v_1, w_1)),$$

so the map T is linear.

The kernel of T is the set of pairs (v, w) with $v \in V, w \in W, v + w = 0$. But if $v + w = 0$, then $v = -w$, so both v and w are in $V \cap W$. Therefore any element of the kernel is a pair $(v, -v), v \in V \cap W$. Applying T, we verify that any such pair is indeed in the kernel. Therefore the kernel of T is exactly the set of pairs $(v, -v), v \in V \cap W$.

The image of T is the set $v + w, v \in V, w \in W$. This is $V + W$, by definition. (Can you see why *any* element of $V + W$ is in the image?).

(c) Given any element f of $\mathcal{L}(V \oplus W, U)$, define $T(f) = (g, h)$, where $g \in \mathcal{L}(V, U)$ is given by $g(v) = f(v, 0)$ and $h \in \mathcal{L}(W, U)$ is given by $h(w) = f(0, w)$. It is easily checked that T is linear.

For any element (g, h) of $\mathcal{L}(V, U) \oplus \mathcal{L}(W, U)$, let $S((g, h)) = f \in \mathcal{L}(V \oplus W, U)$ be given by $f((v, w)) = g(v) + h(w)$. Again, it is easily checked that S is linear and that T and S are inverses of one another, so are isomorphisms. Hence the spaces $\mathcal{L}(V \oplus W, U)$ and $\mathcal{L}(V, U) \oplus \mathcal{L}(W, U)$ are canonically isomorphic, as we did not use any particular basis in our construction of the maps T and S.

(d) Let V be a vector space, and consider the 0–dimensional vector space $\{0\}$. Then the elements of the direct sum $V \oplus \{0\}$ are of the form $(v, 0), v \in V$, and the map taking $(v, 0)$ to v is an isomorphism from $V \oplus \{0\}$ to V. Therefore taking the direct sum with the vector space $\{0\}$ does not change the isomorphism class of the vector space V.

Let U, V and W be any vector spaces over the same field. It is easily checked that the map from $(U \oplus V) \oplus W$ to $U \oplus (V \oplus W)$ given by $T(((u, v), w)) = (u, (v, w))$ is linear and is an isomorphism.

Therefore $(U \oplus V) \oplus W$ and $U \oplus (V \oplus W)$ are isomorphic as vector spaces, so taking direct sums is associative. (Could you check that this map is linear? An isomorphism?)