Math 113 Homework 6

Due Friday, May 17, 2013 by 4 pm

Please remember to write down your name and Stanford ID number, and to staple your solutions. Solutions are due to the Course Assistant, Graham White, in his office, 380-380R (either hand your solutions directly to him or leave the solutions under his door). As usual, please justify all of your solutions and/or answers with carefully written proofs.

Book problems: Solve Axler Chapter 8 problems 22,23,27,29,30 (pages 188-191), Chapter 9 problems 3, 10 (pages 210-211).

1. *Bilinear maps.* Let V, W, and X be vector spaces over $\mathbb{F}$. A function $f : V \times W \to X$ is said to be *bilinear* if it is linear in each variable separately. That is,

$$f(\mathbf{v}, c\mathbf{w} + d\mathbf{w}') = cf(\mathbf{v}, \mathbf{w}) + df(\mathbf{v}, \mathbf{w}')$$

and

$$f(a\mathbf{v} + b\mathbf{v}', \mathbf{w}) = af(\mathbf{v}, \mathbf{w}) + bf(\mathbf{v}', \mathbf{w}).$$

Note that if f is bilinear, then

$$f(\mathbf{v} + \mathbf{v}', \mathbf{w} + \mathbf{w}') \neq f(\mathbf{v}, \mathbf{w}) + f(\mathbf{v}', \mathbf{w}')!$$

Instead,

$$\begin{aligned} f(\mathbf{v} + \mathbf{v}', \mathbf{w} + \mathbf{w}') &= f(\mathbf{v}, \mathbf{w} + \mathbf{w}') + f(\mathbf{v}', \mathbf{w} + \mathbf{w}') \\ &= f(\mathbf{v}, \mathbf{w}) + f(\mathbf{v}, \mathbf{w}') + f(\mathbf{v}', \mathbf{w}) + f(\mathbf{v}', \mathbf{w}') \end{aligned}$$

(a) Let $\mathcal{L}(V \times W, X)$ denote the set of bilinear maps. Prove that $\mathcal{L}(V \times W, X)$ is a vector space. *Hint: it suffices to check that it's a subspace of $X^{V \times W}$, the set of all maps from $V \times W$ to X.*

(b) Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be a basis of V and $\mathbf{w}_1, \ldots, \mathbf{w}_l$ be a basis of W. Prove that a bilinear map $f : V \times W \to X$ is determined uniquely by its values on all pairs of basis elements $(\mathbf{v}_i, \mathbf{w}_j)$. Find a basis for the space of bilinear maps $V \times W \to \mathbb{F}$, and thereby calculate its dimension.

Example: Let $V = \mathcal{P}(\mathbb{R})$ denote the set of polynomials with real coefficients. Let

$$T : V \times V \to \mathbb{R}$$

be the map

$$(p, q) \mapsto \int_{-1}^{1} p(x)q(x)dx.$$

Then, T is bilinear (you should verify this, but do not need to submit it as part of your homework assignment) *Note*: T has the additional property that it is *symmetric bilinear*: $T(p, q) = T(q, p)$.

We will construct many more examples of bilinear maps over the next few weeks.

2. *Complexification.* In class this past week, I alluded to the fact that the classification of real linear operators (from the existence of dimension 1 or 2 invariant subspaces to decomposition theorems and Jordan normal form) could be deduced from the analogous classification of complex linear operators.

Here is a rough idea: if one wants to find eigenvalues and eigenpairs for a real linear operator T, first we should think of it as a "complex linear operator" $T_{\mathbb{C}}$. For example, if we are in $\mathbb{R}^n$, T corresponds to a matrix A in standard coordinates. We could take the same matrix A, now thought of as consisting of complex entries, to obtain a linear operator on $\mathbb{C}^n$ that we can call $T_{\mathbb{C}}$. It will turn out (non-obviously) that the characteristic polynomial of $T_{\mathbb{C}}$ is the same as that of T.

Now, given $T_{\mathbb{C}}$, we can show that its non-real eigenvalues, which are the roots of of its characteristic polynomial, must come in conjugate pairs. We can pair up factors with conjugate roots and multiply them to get the real quadratic polynomials (and hence the *eigenpairs*) appearing in the characteristic polynomial of T. Thus, the eigenvalues of $T_{\mathbb{C}}$ are related to the eigenvalues and eigenpairs of T. For example, if the characteristic polynomial of $T_{\mathbb{C}}$ has complex monomial factors

$$(z-3)(z+i)(z-i)$$

then, by pairing the conjugate roots i and $-i$, we obtain the real factors of the characteristic polynomial of T:

$$(x-3)(x^2+1).$$

Thus, in this case, the real operator T has an eigenvalue 3 and an eigenpair $(0,1)$.

So far, this has been a vague sketch, but we can now begin to make such an approach more precise. The first step is to associate to a real vector space and a real linear operator a natural complex vector space and complex linear operator, which will have the "same" matrix. This is a process known as **complexification**.

Here is a formal definition: Given a real vector space V, we define its **complexification** $V_{\mathbb{C}}$ to be, as a real vector space,

$$V_{\mathbb{C}} := V \oplus V.$$

where $\oplus$ denotes the formal direct sum from last week. To aid in intuition, we refer to an element of the complexification not as $\mathbf{v} \oplus \mathbf{v}'$ but rather as

$$\mathbf{v} \oplus i\mathbf{v}'.$$

We inserted an i above to indicate that the second copy of V should be thought of as the *imaginary part*, and the first copy should be thought of as the *real part.*

Addition in $V_{\mathbb{C}}$ is the usual addition. In order for $V_{\mathbb{C}}$ to be a complex vector space, we must define multiplication by a complex scalar. Define the product

$$(a+bi)\cdot(\mathbf{v} \oplus i\mathbf{v}') := (a\mathbf{v} - b\mathbf{v}') \oplus i(a\mathbf{v}' + b\mathbf{v}).$$

This product agrees with the usual multiplication rules for complex numbers, if we think of the $\mathbf{v}'$ component of $(\mathbf{v} \oplus i\mathbf{v}')$ as imaginary.

(a) Verify that $V_{\mathbb{C}}$ is indeed a vector space over $\mathbb{C}$. (Hint: some of the axioms will hold automatically, because you know that $V_{\mathbb{C}} = V \oplus V$ is already a vector space over $\mathbb{R}$.) Construct an isomorphism $(\mathbb{R}^n)_{\mathbb{C}} \cong \mathbb{C}^n$.

(b) Elements $\mathbf{v} \in V$ give rise to elements in $V_{\mathbb{C}}$ (the corresponding element is $\mathbf{v} \oplus i\mathbf{0}$, but when it is implicit, we will simply also call the associated elements $\mathbf{v}$). Prove that if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis of V (over $\mathbb{R}$), then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is also a basis of $V_{\mathbb{C}}$ (over $\mathbb{C}$). Note that $V_{\mathbb{C}}$ thus has dimension n as a complex vector space (even though its dimension as a real vector space is $2n$).

(c) Now, suppose we are given a linear map $T : V \to V$. Construct a map

$$T_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}$$

extending T (i.e. it agrees with T on the real subspace $V \oplus \{0\}$), and prove that it is linear over $\mathbb{C}$. Prove that

$$\mathcal{M}(T, \underline{\mathbf{v}}) = \mathcal{M}(T_{\mathbb{C}}, \underline{\mathbf{v}})$$

where $\underline{\mathbf{v}}$ on the left is a basis of V, and $\underline{\mathbf{v}}$ is the same basis thought of as a basis of $V_{\mathbb{C}}$. $T_{\mathbb{C}}$ is called the *complexification* of the linear map T.

(d) Recall that complex numbers come equipped with an operation called *conjugation*, denoted by

$$\overline{a + bi} := a - bi.$$

Conjugation distinguishes real elements, the unique elements λ for which $\bar{\lambda} = \lambda$.

We can emulate this operation for the complexification of a vector space, in a manner as follows.
First, map between complex vector spaces

$$T : V \to W$$

is said to be *complex anti-linear* if it is additive, but the homogeneity condition is modified by complex conjugation:

$$T(c \cdot \mathbf{v}) = \bar{c} T\mathbf{w}.$$

Now, given a real vector space V, consider the following map on its complexification $V_{\mathbb{C}}$, which we also call *conjugation*:

$$\begin{aligned} Conj := \overline{(\cdot)} : V_{\mathbb{C}} &\to V_{\mathbb{C}} \\ \mathbf{v} \oplus i\mathbf{v}' &\mapsto \mathbf{v} \oplus i(-\mathbf{v}'). \end{aligned}$$

Prove that this is a complex anti-linear map. If $T : V_{\mathbb{C}} \to V_{\mathbb{C}}$ is any complex linear map, prove that

$$\bar{T} := Conj \circ T \circ Conj : V_{\mathbb{C}} \to V_{\mathbb{C}}$$

is also complex linear (note by definition that $\bar{T}(\mathbf{v}) := \overline{T\overline{\mathbf{v}}}$). If $T_{\mathbb{C}}$ denotes the complexification of the real linear map T, what is the relationship between $\bar{T}_{\mathbb{C}}$ and $T_{\mathbb{C}}$? (you can compare their effect on a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of $V_{\mathbb{C}}$ which comes from one on V).

(e) Now, we can finally put all of this together, and understand the relation to eigenvalues. Let us begin with a linear map

$$T : V \to V$$

on a real, finite-dimensional vector space. Consider the complexified map

$$T_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}},$$

and now suppose

$$\mathbf{w} = \mathbf{v}_1 \oplus i\mathbf{v}_2$$

is an eigenvector of $T_{\mathbb{C}}$ with eigenvalue λ (with both $\mathbf{v}_1 \neq 0$ and and $\mathbf{v}_2 \neq 0$). Prove that

$$\overline{\mathbf{w}} = \mathbf{v}_1 \oplus i(-\mathbf{v}_2)$$

is also an eigenvector of $T_{\mathbb{C}}$, this time with eigenvalue $\overline{\lambda}$. Conclude that if λ is not real, the real subspace

$$U := \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2) \subset V$$

is a 2-dimensional T-invariant subspace of V with associated (real) eigenpair $(-(\lambda + \bar{\lambda}), \lambda\bar{\lambda})$.

Recall: An *eigenpair* (α, β) of T is said to be associated to a real 2-dimensional subspace U if $(T^2 + \alpha T + \beta I)|_U = 0)$.

Hint: (1) U arises as the real part of $\operatorname{span}(\mathbf{w}, \overline{\mathbf{w}})$. (2) Note that on the level of complexified operators,

$$(T_{\mathbb{C}}^2 - (\lambda + \bar{\lambda})T_{\mathbb{C}} + (\lambda\bar{\lambda})I) = (T_{\mathbb{C}} - \lambda I)(T_{\mathbb{C}} - \bar{\lambda} I).$$

(f) Why must $\mathbf{v}_1$ and $\mathbf{v}_2$ both not equal zero if λ is not real? If λ is a real eigenvalue of $T_{\mathbb{C}}$, and $\mathbf{w} = \mathbf{v}_1 \oplus i\mathbf{v}_2$ is the associated eigenvector, prove that there exists a λ eigenvector of T. (Hint: There are two cases. The first is if one of $\mathbf{v}_1$ or $\mathbf{v}_2$ equals 0. The second case is if both are non-zero, in which case you can try to apply the previous section).

Conclude that because $T_{\mathbb{C}}$ always has an eigenvalue, T always has a dimension 1 or 2 invariant subspace.

Remark: An identical conjugate-pairs correspondence holds for generalized eigenvectors, for the same reason. Thus, generalized eigenspaces for non-real eigenvalues come in conjugate pairs. One consequence is that the decomposition theorem for real operators (into generalized eigenspaces and generalized eigen-pair spaces) can be deduced from that for complex operators.

Also, this will imply that the characteristic polynomial of T (as a real linear map) is equal to the characteristic polynomial of $T_{\mathbb{C}}$. Going even further, one could use the existence of a Jordan basis for $T_{\mathbb{C}}$ to deduce the corresponding result for the real case.