Math 113 Homework 7

Due Friday, May 24, 2013 by 4 pm

Please remember to write down your name and Stanford ID number, and to staple your solutions. Solutions are due to the Course Assistant, Graham White, in his office, 380-380R (either hand your solutions directly to him or leave the solutions under his door). As usual, please justify all of your solutions and/or answers with carefully written proofs.

Book problems: Solve Axler Chapter 6 problems 10, 12, 17, 20, 24, 25, 26 (page 122-125).

1. Let $(V, \langle \cdot, \cdot \rangle)$ be a finite dimensional inner product space over $\mathbb{F}$ (either $\mathbb{R}$ or $\mathbb{C}$). If we are given a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$, let $g_1, \ldots, g_n \in V^*$ be the functions $g_i : V \to \mathbb{F}$ defined by

$$g_i(\mathbf{v}) = \langle \mathbf{v}, \mathbf{v}_i \rangle.$$

(a) Prove that g_i is a basis for V^*.

(b) Recall that the dual basis $\mathbf{v}_1^*, \ldots, \mathbf{v}_n^*$ is given by

$$\mathbf{v}_j^*(\mathbf{v}_i) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{1}$$

Prove that the basis $(\mathbf{v}_1^*, \ldots, \mathbf{v}_n^*)$ is equal to the basis $(g_1, \ldots, g_n)$ if and only if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is an *orthonormal basis* for V.

2. *Orthonormal lists in infinite dimensions.* Let $V = \mathcal{C}^0([-\pi, \pi], \mathbb{R})$ denote the vector space of continuous functions from the interval $[-\pi, \pi]$ to $\mathbb{R}$. Equip V with the inner product

$$\langle p, q \rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} p(x)q(x)dx$$

Show that the infinite set

$$\{\frac{1}{\sqrt{2}}, \cos x, \sin x, \cos 2x, \sin 2x, \cos 3x, \sin 3x, \ldots, \}$$

is an orthonormal list with respect to this inner product. (The identities

$$\sin kx \sin mx = \frac{1}{2}(\cos(k-m)x - \cos(k+m)x)$$
$$\cos kx \cos mx = \frac{1}{2}(\cos(k-m)x + \cos(k+m)x)$$
$$\sin kx \cos mx = \frac{1}{2}(\sin(k-m)x + \sin(k+m)x)$$

may be helpful).

Remark: Of course, this set is not an *orthonormal basis*, as finite linear combinations of this set do not span V. However, it is a deep theorem in *Fourier analysis* that in fact, certain *convergent infinite linear combinations* do span V! Namely, any continuous function $f \in V$ can be written as an infinite convergent sum

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_m a_m \cos mx + \sum_n b_n \sin nx.$$

Moreover, in this sum, the coefficient of a given orthonormal list element, is given by the usual projection formula, e.g., the coefficient a_m of $\cos mx$ is

$$a_m = \langle f, \cos mx \rangle.$$

3. *Abstract operations on vector spaces II: Tensor products.* Given a pair of vector spaces V and W, last week we defined their *formal direct sum*

$$V \oplus W,$$

This week we will have defined a *formal product* of vector spaces, known as the tensor product and denoted by

$$V \otimes W.$$

One heuristic property first, to motivate the definition: in the same manner that direct sums are additive in dimension, tensor products will be multiplicative in dimension.

Onto the definition: $V \otimes W$ is defined to be the set of elements of the form

$$\sum_k a_k \mathbf{v}_k \otimes \mathbf{w}_k,$$

where a_k is a scalar, $\mathbf{v}_k \in V$, and $\mathbf{w}_k \in W$, and the sum is a finite sum ($\otimes$ is just a formal symbol used to denote the concatenation of $\mathbf{v}_k$ and $\mathbf{w}_k$ in this context). Moreover in $V \otimes W$, there are relations, which give us rules for simplifying expressions. That is, the following expressions are equal:

$$\begin{aligned} a(\mathbf{v} \otimes \mathbf{w}) &= (a\mathbf{v}) \otimes \mathbf{w} = \mathbf{v} \otimes (a\mathbf{w}), \\ (\mathbf{v} + \mathbf{v}') \otimes \mathbf{w} &= \mathbf{v} \otimes \mathbf{w} + \mathbf{v}' \otimes \mathbf{w} \\ \mathbf{v} \otimes (\mathbf{w} + \mathbf{w}') &= \mathbf{v} \otimes \mathbf{w} + \mathbf{v} \otimes \mathbf{w}'. \end{aligned} \tag{2}$$

The zero element is $\mathbf{0}_V \otimes \mathbf{0}_W$, addition is just given by formally adding two finite sums together (and simplifying if possible by using the relations above), and scalar multiplication is as one might expect:

$$a \cdot \sum_k a_k \mathbf{v}_k \otimes \mathbf{w}_k := \sum_k a(a_k \mathbf{v}_k \otimes \mathbf{w}_k)$$

Elements of the form $\mathbf{v} \otimes \mathbf{w}$ are called **pure tensors**. A general element of $V \otimes W$ will not be a pure tensor, but rather a sum of such terms. There are a few differences to note from formal direct sums: note first that

$$\mathbf{v} \otimes \mathbf{w} + \mathbf{v}' \otimes \mathbf{w}' \neq (\mathbf{v} + \mathbf{v}') \otimes (\mathbf{w} + \mathbf{w}');$$

in fact

$$(\mathbf{v} + \mathbf{v}') \otimes (\mathbf{w} + \mathbf{w}') = \mathbf{v} \otimes \mathbf{w} + \mathbf{v} \otimes \mathbf{w}' + \mathbf{v}' \otimes \mathbf{w} + \mathbf{v}' \otimes \mathbf{w}',$$

which is to say that the formal symbol "$\otimes$" acts much like a product, in its distributivity properties.

In a similar vein, note that, using the relations above,

$$\mathbf{0}_V \otimes \mathbf{w} = (0 \cdot \mathbf{0}_V) \otimes \mathbf{w} = \mathbf{0}_V \otimes 0\mathbf{w} = \mathbf{0}_V \otimes \mathbf{0}_W = \mathbf{0}_{V\otimes W},$$

another property formally similar to multiplication. Similarly, $\mathbf{v} \otimes \mathbf{0}_W = \mathbf{0}_{V\otimes W}$.

(a) There is a natural map

$$\phi : V \times W \to V \otimes W$$
$$(\mathbf{v}, \mathbf{w}) \mapsto \mathbf{v} \otimes \mathbf{w}.$$

Prove that this is a *bilinear* map, where *bilinear maps* were defined on HW6.

(b) In fact ϕ is the *universal bilinear map*, as we will see in this exercise. Namely, prove the following: if

$$T : V \times W \to X$$

is a *bilinear map*, in the sense of homework 6, prove that there exists a unique *linear map*

$$\underline{T} : V \otimes W \to X$$

such that $T = \underline{T} \circ \phi$. That is, T factors uniquely as

$$V \times W \xrightarrow{\phi} V \otimes W \xrightarrow{\underline{T}} X$$

where $\underline{T}$ is a linear map.

Hint: What does it mean for a map $\underline{T} : V \otimes W \to X$ to be a linear? Firstly, it means that on a formal sum

$$\sum_k a_k \mathbf{v}_k \otimes \mathbf{w}_k$$

$\underline{T}$ is additive and homogenous, so

$$\underline{T}(\sum_k a_k \mathbf{v}_k \otimes \mathbf{w}_k) = \sum_k a_k \underline{T}(\mathbf{v}_k \otimes \mathbf{w}_k).$$

It also means that $\underline{T}$ should not be affected by applying the relations (2); that is, $\underline{T}((\mathbf{v} + \mathbf{v}') \otimes \mathbf{w})$ should be equal to $\underline{T}(\mathbf{v} \otimes \mathbf{w} + \mathbf{v}' \otimes \mathbf{w})$ and so on.

(c) Prove that $\dim(V \otimes W) = (\dim V) \cdot (\dim W)$ (Hint: given a basis $(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ of V and a basis $\mathbf{w}_1, \ldots, \mathbf{w}_l$ of W, prove that the collection of pure tensors $\{\mathbf{v}_i \otimes \mathbf{w}_j | 1 \leq i \leq k, 1 \leq j \leq l\}$ is a basis of $V \otimes W$).

Crucial Hint: Proving that these elements span is relatively straightforward. However, proving that this collection is linearly independent directly may be a bit more tricky (given how many ways one can try to simplify and expand expressions using the relations (2)). Here is a suggested shortcut: suppose there is a linear relationship among the $\{\mathbf{v}_i \otimes \mathbf{w}_j\}$, so some sum of them with coefficients a_{ij} is 0. Recall on HW6 you found bilinear maps $f_{ij} : V \times W \to \mathbb{F}$ that were 1 on the input $\mathbf{v}_i, \mathbf{w}_j$ and zero on other pairs $\mathbf{v}_s, \mathbf{w}_t$. Apply part (b) to obtain a linear map $\overline{f}_{ij} : V \otimes W \to \mathbb{F}$. What happens if you apply this $\overline{f}_{ij}$ to the linear relationship?

(d) *An example.* Let $\mathbb{F}[x]$ denote the vector space of polynomials in a variable x (which we normally call $\mathcal{P}(\mathbb{F})$), and let $\mathbb{F}[y]$ denote the vector space of polynomials in a variable y (this is the same vector space, where we've relabled the variable). Construct an isomorphism

$$\mathbb{F}[x] \otimes \mathbb{F}[y] \xrightarrow{\sim} \mathbb{F}[x, y]$$

where $\mathbb{F}[x, y]$ is a new vector space: the vector space of *polynomials in two variables*; i.e. finite sums of the form

$$\sum_{i \geq 0, j \geq 0} a_{ij} x^i y^j$$

(e) The tensor product functions much like a product on vector spaces (minus the existence of multiplicative inverses!) That is, up to canonical isomorphism it distributes with formal direct sum,

$$V \otimes (W \oplus X) \cong (V \otimes W) \oplus (V \otimes X) \tag{3}$$

there is a multiplicative identity

$$V \otimes \mathbb{F} \cong \mathbb{F} \otimes V \cong V \tag{4}$$

and multiplying by 0 (the additive identity) results in 0:

$$V \otimes \{0\} \cong \{0\} \otimes V \cong \{0\}. \tag{5}$$

Prove most of these facts. More precisely, construct and verify the canonical isomorphisms (4) and (5), and construct a canonical map for (3) (no need to verify it's an isomorphism).

Remark: For those interested in applications, a brief motivation: Tensor products have many incarnations in applied fields, especially all over physics. If one likes thinking in terms of coordinates, the tensor product $\mathbb{R}^m \otimes \mathbb{R}^n$ can be identified with $m \times n$ matrices, and higher tensor products $\mathbb{R}^{m_1} \otimes \cdots \otimes \mathbb{R}^{m_k}$ can be identified with $m_1 \times \cdots \times m_k$ multi-dimensional arrays of scalars.