Math 113 — Homework 7

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Book problems

10. The first element of our basis is the function 1.

    The second element is proportional to $x - \langle x, 1\rangle 1 = x - \frac{1}{2}$. Normalising, we get $\sqrt{12}(x - \frac{1}{2})$.

    The third element is proportional to

$$\begin{aligned} x^2 - \langle x^2, 1\rangle 1 - \langle x^2, \sqrt{12}(x - \frac{1}{2})\rangle (x - \frac{1}{2}) &= x^2 - \frac{1}{3} - \sqrt{12}(\frac{1}{4} - \frac{1}{6})\sqrt{12}(x - \frac{1}{2}) \\ &= x^2 - \frac{1}{3} - (x - \frac{1}{2}) \\ &= x^2 - x + \frac{1}{6} \end{aligned}$$

    Normalising, we get $\sqrt{5}(6x^2 - 6x + 1)$.

12. If e_1 is a vector with $\mathrm{Span}(e_1) = \mathrm{Span}(v_1)$, then e_1 is a scalar multiple of v_1. Let $e_1 = cv_1$. Then $\langle e_1, e_1\rangle = |c|^2\langle v_1, v_1\rangle$, which is equal to 1 exactly when $|c| = \pm\frac{1}{\sqrt{\langle v_1, v_1\rangle}}$. Hence there are two choices for e_1. For the remainder of the argument, we will only be concerned with the span of e_1 and whether vectors are orthogonal to it, so it doesn't matter which choice we make.

    Now, orthogonal complement to $\mathrm{Span}(e_1)$ inside $\mathrm{Span}(v_1, v_2)$ is one dimensional. As before, there are two choices of vector e_2 which are contained in this subspace with $\langle e_2, e_2\rangle = 1$. We now repeat this argument. There are two choices for e_3 in the orthogonal complement of $\mathrm{Span}(e_1, e_2)$ inside $\mathrm{Span}(v_1, v_2, v_3)$, etc.

    For each e_i, there were two possibilities, and these choices could be made independently. Therefore there are 2^n possible lists $\{e_1, e_2, \ldots, e_n\}$ satisfying the desired condition.

17. Any element of V can be written as $v = (v - P(v)) + P(v)$. We have that $(v - P(v)) \in \ker(P)$, because $P^2 = P$, and that $P(v) \in \mathrm{im}(P)$ by definition. Therefore $\ker(P)$ and $\mathrm{im}(P)$ span V. Every element of $\ker(P)$ is orthogonal to every element of $\mathrm{im}(P)$ and a vector is orthogonal to itself only if it is the zero vector, so $\ker(P) \cap \mathrm{im}(P) = \{0\}$. Therefore $V = \ker(P) \oplus \mathrm{im}(P)$.

    Using the above decomposition, the component of v in $\mathrm{im}(P)$ is $P(v)$, so P is the orthogonal projection onto $\mathrm{im}(P)$.

20. If U and $U^\perp$ are both invariant under T, then let v be an arbitrary element of V. We may write $v = u + u'$, $u \in U$ and $u' \in U^\perp$. Then we have that

$$\begin{aligned} P_U T(v) &= P_U T(u + u') \\ &= P_U(T(u) + T(u')) \\ &= T(u) \quad (T(u) \in U, T(u') \in U^\perp) \\ &= T(P_U(u + u')) & = TP_U(v) \end{aligned}$$

    Therefore $P_U T = TP_U$.

Now, assume that $P_U T = T P_U$. Let u be any element of U. We have that $P_U T(u) = T P_U(u)$, so $P_U T(u) = T(u)$. Therefore $T(u)$ is fixed by P_U, so is an element of U. Hence U is invariant under T. We have that $\mathrm{Id}_V = P_U + P_{U^\perp}$, and Id_V and P_U both commute with T, so $P_{U^\perp}$ commutes with T. Therefore the same argument shows that $U^\perp$ is invariant under T.

We have shown that U and $U^\perp$ are invariant under T if and only if $P_U T = T P_U$.

24. The required condition is linear in p, so it suffices to give a q that works for $p = 1, p = x$ and $p = x^2$. That is, we need the following three conditions:

$$\int_0^1 q(x)dx = 1$$
$$\int_0^1 xq(x)dx = \frac{1}{2}$$
$$\int_0^1 x^2 q(x)dx = \frac{1}{4}$$

Let $q(x) = ax^2 + bx + c$. Our equations reduce to

$$\frac{a}{3} + \frac{b}{2} + c = 1$$
$$\frac{a}{4} + \frac{b}{3} + \frac{c}{2} = \frac{1}{2}$$
$$\frac{a}{5} + \frac{b}{4} + \frac{c}{3} = \frac{1}{4}$$

Solving these gives $a = -15, b = 15, c = -\frac{3}{2}$. Therefore the polynomial $q(x)$ is $q(x) = -15x^2 + 15x - \frac{3}{2}$.

25. Proceeding as in the previous part, we have the equations

$$\int_0^1 q(x)dx = \int_0^1 \cos(\pi x)dx$$
$$\int_0^1 xq(x)dx = \int_0^1 x\cos(\pi x)dx$$
$$\int_0^1 x^2 q(x)dx = \int_0^1 x^2\cos(\pi x)dx$$

Using integration by parts, we evaluate these as

$$\int_0^1 q(x)dx = 0$$
$$\int_0^1 xq(x)dx = -\frac{2}{\pi^2}$$
$$\int_0^1 x^2 q(x)dx = -\frac{2}{\pi^2}$$

As before, let $q(x) = ax^2 + bx + c$. We get that

$$\frac{a}{3} + \frac{b}{2} + c = 0$$
$$\frac{a}{4} + \frac{b}{3} + \frac{c}{2} = -\frac{2}{\pi^2}$$
$$\frac{a}{5} + \frac{b}{4} + \frac{c}{3} = -\frac{2}{\pi^2}$$

Solving these gives $a = 0, b = -\frac{24}{\pi^2}, c = \frac{12}{\pi^2}$. Therefore the polynomial $q(x)$ is $q(x) = -\frac{24}{\pi^2}x + \frac{12}{\pi^2}$.

26. Let an orthonormal basis for V be $\{e_1, \ldots, e_n\}$. We have that for each $v \in V$, $\langle Tv, a\rangle = \langle v, T^*a\rangle$. (Note that the first inner product is in V and the second is in $\mathbb{F}$). Taking v to be e_i, we get that for each i, $\langle Te_i, a\rangle = \langle e_i, T^*a\rangle$. But now we have an expression for $\langle e_i, T^*a\rangle$ for each e_i, which we can use to reconstitute T^*a. We get that

$$T^*a = \langle a, Te_1\rangle e_1 + \ldots + \langle a, Te_n\rangle e_n.$$

(The order of the elements $\langle a, b\rangle$ vs $\langle b, a\rangle$ is important, but reversing it just results in complex conjugation).

Other problems:

1. (a) Assume that some linear combination $a_1g_1 + \ldots + a_ng_n = 0$. Then for any $v \in V$, we have that

$$a_1\langle v, v_1\rangle + \ldots + a_n\langle v, v_n\rangle = 0$$

Therefore

$$\langle v, \overline{a_1}v_1 + \ldots + \overline{a_n}v_n\rangle = 0.$$

This equation is true for all $v \in V$, including $v = \overline{a_1}v_1 + \ldots + \overline{a_n}v_n$. But the dot product of a vector with itself is zero only if that vector is zero. Therefore $\overline{a_1}v_1 + \ldots + \overline{a_n}v_n = 0$. But $v_1, \ldots, v_n$ is linearly independent, so each $\overline{a_i}$ is zero. Therefore each a_i is zero, and we have shown that $g_1, \ldots, g_n$ are linearly independent.

We know that the dimension of V^* is n, so any set of n linearly independent vectors is a basis. Therefore $g_1, \ldots, g_n$ is a basis of V^*.

(b) The basis $(v_1^*, \ldots, v_n^*)$ being equal to the basis $(g_1, \ldots, g_n)$ is equivalent to the statement that $\langle v_i, v_j\rangle = \delta_{ij}$ for each i and j, by the definition of v_i^*. But this is the definition of orthonormality of a basis.

2. In order to show that this list is orthonormal, we need to show that the inner product of any element with itself is one and that any two elements are orthogonal. We check these as follows, where k and l are any positive integers, $k \neq l$.

$$
\begin{aligned}
\langle\frac{1}{\sqrt(2)},\frac{1}{\sqrt(2)}\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}dx \\
&= \frac{1}{\pi}(\frac{\pi}{2}-\frac{-\pi}{2}) \\
&= 1 \\
\langle\cos(kx),\cos(kx)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2(kx)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos(2kx)+1}{2}dx \\
&= \frac{1}{\pi}\left[\frac{\sin(2kx)}{2k}+\frac{x}{2}\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi}(\frac{\pi}{2}-\frac{-\pi}{2}) \\
&= 1 \\
\langle\sin(kx),\sin(kx)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi}\sin^2(kx)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1-\cos(2kx)}{2}dx \\
&= \frac{1}{\pi}\left[\frac{x}{2}-\frac{\sin(2kx)}{2k}\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi}(\frac{\pi}{2}-\frac{-\pi}{2}) \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\langle \frac{1}{\sqrt{2}}, \cos(kx)\rangle &= \frac{1}{\sqrt{2}}\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(kx)dx \\
&= \frac{1}{2\pi}\left[\frac{\sin(kx)}{k}\right]_{-\pi}^{\pi} \\
&= 0 \\
\langle \frac{1}{\sqrt{2}}, \sin(kx)\rangle &= \frac{1}{\sqrt{2}}\frac{1}{\pi}\int_{-\pi}^{\pi}\sin(kx)dx \\
&= \frac{-1}{2\pi}\left[\frac{\cos(kx)}{k}\right]_{-\pi}^{\pi} \\
&= 0 \\
\langle \cos(kx), \cos(lx)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(kx)\cos(lx)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos(kx+lx)+\cos(kx-lx)}{2}dx \\
&= 0 \quad \text{(as calculated above, as } k+l \text{ and } k-l \text{ are nonzero)} \\
\langle \sin(kx), \sin(lx)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi}\sin(kx)\sin(lx)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos(kx+lx)-\cos(kx-lx)}{2}dx \\
&= 0 \quad \text{(as calculated above, as } k+l \text{ and } k-l \text{ are nonzero)} \\
\langle \cos(kx), \sin(lx)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(kx)\sin(lx)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin(kx+lx)-\sin(kx-lx)}{2}dx \\
&= 0 \quad \text{(as calculated above, as } k+l \text{ and } k-l \text{ are nonzero)}
\end{aligned}
$$

Therefore the given set of functions is orthonormal.

3. (a) We check that ϕ is bilinear by using the relations used in the definition of the tensor product, as follows

$$
\begin{aligned}
\phi(av+bv', w) &= (av+bv')\otimes w \\
&= av\otimes w + bv'\otimes w \\
&= a(v\otimes w) + b(v'\otimes w) \\
&= a\phi(v,w) + b\phi(v',w)
\end{aligned}
$$

Linearity in the second variable is checked via a similar calculation.

(b) We define $\underline{T}(v\otimes w) = T(v,w)$ for each $v \in V$ and each $w \in W$, and extend this definition to all of $V\otimes W$ linearly. The function $\underline{T}$ is well-defined because T is bilinear. (Could you write down the calculations that would verify this?). By construction, we have that $T = \underline{T}\circ\phi$. Therefore there exists at least one map $\underline{T}$ with the required properties.

Assume there were two linear maps $\underline{T}, \underline{T}'$ with $T = \underline{T}\circ\phi$ and $T = \underline{T}'\circ\phi$. Then $\underline{T}$ and $\underline{T}'$ would be equal on the image of ϕ. But this means that $\underline{T}$ and $\underline{T}'$ are equal on any pure tensor, and any element of $V\otimes W$ is a linear combination of pure tensors. Therefore $\underline{T} = \underline{T}'$, so there is a unique linear map with this property.

(c) Any element of $V\otimes W$ is a linear combination of pure tensors. Using the bilinearity of the tensor product, we may write any pure tensor $v\otimes w$ as a linear combination of putative basis elements $v_i\otimes w_j$ by expanding v and w in terms of the bases of V and W. Therefore the elements $v_i\otimes w_j$ span $V\otimes W$.

As in the hint, assume that there is some linear dependence relation between the $v_i \otimes w_j$, with coefficients a_{ij}. The function $\underline{f_i j}$ is equal to 1 on $v_i \otimes w_j$ and is zero on all other elements of our supposed basis (Why? Check this using part b)). Applying $\underline{f_i j}$ to our linear dependence relation gives that $a_{ij} = 0$. We repeat this for each pair (i, j), showing that each $a_{ij} = 0$. Therefore the $v_i \otimes w_j$ are linearly independent.

We have shown that the set of $v_i \otimes w_j$ spans $V \otimes W$ and is linearly independent, so it is a basis. Therefore the dimension of the $V \otimes W$ is the product of the dimensions of V and of W, by counting the elements of this basis.

(d) Consider the map ϕ defined by $\phi(p(x), q(y)) = p(x)q(y)$ and extended linearly to all of $\mathbb{F}[x] \otimes \mathbb{F}[y]$. The function ϕ is linear because polynomial multiplication is bilinear. Define another map ϕ^{-1} by $\phi^{-1}(x^a y^b) = x^a \otimes y^b$, extended linearly to $\mathbb{F}[x, y]$. These maps are inverses of one another, which is easily verified on monomials. (Why is it enough to check this on monomials?). Therefore ϕ is an isomorphism from $\mathbb{F}[x] \otimes \mathbb{F}[y]$ to $\mathbb{F}[x, y]$, as required.

(e) For (3), the canonical isomorphism is the linear map which takes $v \otimes (w, x)$ to $(v \otimes w, v \otimes x)$.

For (4), note that for any $v \in V$ and $a \in \mathbb{F}$, we have that $v \otimes a = a(v \otimes 1)$. Therefore, let ϕ be the linear map defined by $\phi(v \otimes a) = av$ and its inverse be $\phi^{-1}(v) = v \otimes 1$. It is easily checked that these maps linear and are inverses of one another (on pure tensors in $V \otimes \mathbb{F}$ and on any element of V). A similar argument works for $\mathbb{F} \otimes V$.

For (5), note that for any $v \in V$, we have that $v \otimes 0 = 0$. Therefore, the vector space $V \otimes \{0\}$ is the unique vector space with only one element. There is only one map between vector spaces which each have one element. This map is trivially linear and bijective, thus is an isomorphism. Likewise for $0 \otimes V$.