Math 113 Homework 8

Due Thursday, May 30, 2013 by 4 pm (Note earlier deadline!)

Please remember to write down your name and Stanford ID number, and to staple your solutions. Solutions are due to the Course Assistant, Graham White, in his office, 380-380R (either hand your solutions directly to him or leave the solutions under his door). As usual, please justify all of your solutions and/or answers with carefully written proofs.

Book problems: Solve Axler Chapter 7 problems 1, 4, 8, 10, 14, 15, 32 (the last problem will require some knowledge of Singular Value Decomposition, which we will do in class Tuesday).

1. *Tensor products and linear maps.* Though their definition may seem strange at first, tensor products are very naturally related to spaces of linear maps. This is maybe not that suprising in light of the fact, e.g., that both $V \otimes W$ and $\mathcal{L}(V, W)$ have dimension $(\dim V) \cdot (\dim W)$. In this exercise, we have two goals: to explore this fundamental relationship, and to use it to define the *trace* of a linear operator.

You may find our class notes on the tensor product, available on the course webpage, useful. For the next two parts, you should first look over Section 1 of the notes, particularly how to construct linear maps out of the tensor product. With the notes in hand, this exercise should (hopefully) be more straightforward.

(a) *Evaluation maps.* Given a vector space V, let V^* denote its linear dual. Consider the map

$$\begin{aligned} ev : V^* \times V &\to \mathbb{F} \\ (F, \mathbf{v}) &\mapsto F(\mathbf{v}). \end{aligned}$$

Prove that ev is a bilinear map, and thus induces a map (which we'll also call by the same name)

$$ev : V^* \otimes V \to \mathbb{F}.$$

If $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is a basis of V and $(\mathbf{v}_1^*, \ldots, \mathbf{v}_n^*)$ the corresponding dual basis, what are the values of ev on the basis elements $\mathbf{v}_i^* \otimes \mathbf{v}_j$?

Note: This is related to an isomorphism from V to $(V^*)^*$ which we constructed on HW4. Namely, we constructed that map as

$$E : V \to \mathcal{L}(V^*, \mathbb{F}), \text{ so } E \in \mathcal{L}(V, \mathcal{L}(V^*, \mathbb{F}))$$

by sending $\mathbf{v}$ to $\mathrm{eval}_{\mathbf{v}}$, the functional on V^* that takes a functional F and returns $F(\mathbf{v})$. Given such a map, we can associate a bilinear map

$$\tilde{E} : V \times V^* \to \mathbb{F}$$

by

$$\tilde{E}(\mathbf{v}, F) = E(\mathbf{v})(F)$$

(Note that this map is bilinear!). $\tilde{E}$ is the same map as ev, with the inputs reversed. This is part of a more general phenomenon: there are canonical isomorphisms

$$\mathcal{L}(V, \mathcal{L}(W, X)) \cong \mathcal{L}(V \times W, X) \cong \mathcal{L}(V \otimes W, X).$$

The first isomorphism is obtained by sending T to the bilinear map $\tilde{T}(\mathbf{v}, \mathbf{w}) = T(\mathbf{v})(\mathbf{w})$, and the second is one we studied last week.

(b) Given a pair of vertor spaces V and W, construct a *canonical* linear map

$$V^* \otimes W \xrightarrow{\alpha} \mathcal{L}(V, W) \tag{1}$$

When V and W are finite-dimensional, prove that this map is in fact an isomorphism (This requires using bases). **Hint**: it suffices to check that the image of a basis on the left is a basis on the right.

(c) *Trace*. Given an $n \times n$ matrix A, in a different class you may have seen that there is a fundamental scalar, called the *trace*, that one can associate to the matrix A:

$$\mathrm{tr}(A) := \sum_i a_{ii}.$$

In words, the trace of a matrix A is the sum of its diagonal values.

Now, given a linear transformation $T : V \to V$ where V is an n-dimensional vector space, one can pick a basis $\underline{\mathbf{v}} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ to obtain a matrix

$$\mathcal{M}(T, \underline{\mathbf{v}}).$$

One might be tempted to define the *trace of* T as

$$\mathrm{tr}(T) := \mathrm{tr}(\mathcal{M}(T, \underline{\mathbf{v}})),$$

but a priori, this may depend on the matrix we pick for T! In other words, it's not at all clear that this definition really gives a number that depends only on T, and not on the particular basis we used to write a matrix for T.

Let us give a definition that is manifestly independent of basis. If V is finite-dimensional, then from part b, the canonical map

$$\alpha : V^* \otimes V \to \mathcal{L}(V, V)$$

in (1) is an isomorphism. Thus any linear operator

$$T : V \to V$$

corresponds via taking α^{-1} to an element

$$\alpha^{-1}(T) \in V^* \otimes V.$$

This is the inverse of a canonical map, hence is canonical (note: α^{-1} may require choosing a basis to write explicitly, but the result is independent of choices).

Now, from part (a), there is a canonical map

$$ev : V^* \otimes V \longrightarrow \mathbb{F}.$$

Thus, we can define the *trace of* T to be the scalar

$$tr(T) := ev(\alpha^{-1}(T)),$$

which is by construction independent of choices of bases. As it is defined as a composition of linear maps, tr is a linear map from $\mathcal{L}(V,V)$ to $\mathbb{F}$, so $tr(aS+bT) = a\cdot tr(S) + b\cdot tr(T)$.

Prove that if $\mathcal{M}(T,\underline{\mathbf{v}})$ is the matrix of T with respect to any basis, then the *trace of* T equals the usual trace of the matrix.

Hint: To start, given $T : V \to V$, and a basis $\mathbf{v}_1,\ldots,\mathbf{v}_n$ of V and the associated dual basis $(\mathbf{v}_1^*,\ldots,\mathbf{v}_n^*)$ of V^*, write the element of $V^*\otimes V$ that corresponds to T in terms of the basis $\mathbf{v}_i^*\otimes\mathbf{v}_j$. (this should have some relationship to the matrix coefficients of T with respect to V). Next, determine what ev does to basis elements $\mathbf{v}_i^*\otimes\mathbf{v}_j$ and thus what it does to T.
As a consequence, we see that traces of similar matrices are equal.

(d) The trace is not multiplicative, in that for general linear maps

$$tr(ST) \neq tr(S)tr(T).$$

However, show that

$$tr(ST) = tr(TS).$$

Hint: This is probably simplest to check by computing using matrices with respect to a given basis.

(Careful! If you're ever using this in the future, this does not for example imply that $tr(STR) = tr(SRT)$. One can only reverse the order of a pair of linear maps at a time, so, e.g. $tr(STR) = tr((ST)R) = tr(R(ST)) = tr(RST)$).

2. Extra Credit: *An inner product on linear maps.*
Let $(V,\langle\cdot,\cdot\rangle)$ be a finite-dimensional inner product space. For linear operators $S,T\in\mathcal{L}(V)$, let $\underline{e} = (e_1,\ldots,e_n)$ be an orthonormal basis of V, and define

$$\langle S,T\rangle = \sum_{i=1}^{n}\langle Se_i, Te_i\rangle. \tag{2}$$

Prove that $\langle\cdot,\cdot\rangle$ gives an inner product on $\mathcal{L}(V)$, and that it does not depend on the choice of orthonormal basis $\underline{e}$. That is, if $\underline{f} = (f_1,\ldots,f_n)$ is another orthonormal basis of V, then (2), with e_i's replaced by f_i's gives the same result. Finally, prove that that the inner product satisfies $\langle S,T\rangle = \langle T^*,S^*\rangle$.

Crucial hint: You should not have to do much to prove $\langle S,T\rangle$ is independent of choice of orthonormal basis if you can write this inner product as the trace of a linear map…(then by the last problem, such a definition will automatically be independent of choice of basis).