Math 113 — Homework 8

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Book problems

1. From the previous homework set, we know that an orthonormal basis for $P_2(\mathbb{R})$ with the given inner product is $\{1, \sqrt{3}(2x-1), \sqrt{5}(6x^2-6x+1)\}$. The matrix of T with respect to this basis is

$$\begin{pmatrix} 0 & \sqrt{3} & -3\sqrt{5} \\ 0 & 1 & -\sqrt{15} \\ 0 & 0 & 0 \end{pmatrix}$$

   This matrix is not Hermitian, so T is not self-adjoint. An operator is self-adjoint if and only if its matrix with respect to any orthonormal basis is Hermitian. The basis $\{1, x, x^2\}$ is not orthonormal, so the matrix of T with respect to this basis being Hermitian is unimportant.

4. If P is an orthogonal projection, then $V = \operatorname{im}(P) \oplus \ker(P)$. Consider the matrix of P with respect to a basis of V which is the union of bases for $\operatorname{im}(P)$ and for $\ker(P)$. The operator P acts as the identity on $\operatorname{im}(P)$ and as the zero operator on $\ker(P)$, so this matrix is diagonal, with ones and zeros on the diagonal. Such a matrix is Hermitian, so P is self-adjoint.

   If P is self-adjoint, consider an orthonormal basis for $\operatorname{im}(P)$ and extend it to an orthonormal basis of V. The matrix of P with respect to this basis may be written as a block matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

   where A is the matrix of P acting on $\operatorname{im}(V)$ with respect to the chosen basis of that space.

   We know that P acts on $\operatorname{im}(P)$ as the identity, so A is an identity matrix. The image under P of any element of V lies in $\operatorname{im}(P)$, so C and D are both comprised solely of zeros. Finally, M is Hermitian, so B is the Hermitian conjugate of C, and is hence also comprised of zeros. The matrix of P with respect to an orthonormal basis is diagonal with all entries ones and zeros, so P is an orthonormal projection.

8. Consider the standard inner product on $\mathbb{R}^3$. Then $(1, 2, 3)$ is a 0–eigenvector of T and $(2, 5, 7)$ is a 1–eigenvector of T. These two vectors are not orthogonal, violating Corollary 7.14 of Axler.

10. From Theorem 7.9 of Axler, there is an orthonormal basis of V comprised of eigenvectors of T. If $T^9 = T^8$, then each eigenvector λ of T satisfies $\lambda^9 = \lambda^8$, so $\lambda = 0$ or $\lambda = 1$. From Corollary 7.14 of Axler, the result follows.

14. The operator T is self-adjoint, so there is an orthonormal basis of V consisting of eigenvectors of T. Let this basis be $v_1, \ldots, v_n$, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $v = a_1 v_1 + \ldots a_n v_n$. We know that $a_1^2 + \ldots + a_n^2 = 1$. We calculate the following

$$\begin{aligned} ||Tv - \lambda v||^2 &= ||(\lambda_1 - \lambda)a_1 v_1 + \ldots + (\lambda_n - \lambda)a_n v_n||^2 \\ &= ||(\lambda_1 - \lambda)a_1 v_1||^2 + \ldots + ||(\lambda_n - \lambda)a_n v_n||^2 \\ &= |(\lambda_1 - \lambda)a_1|^2 + \ldots + |(\lambda_n - \lambda)a_n|^2 \end{aligned}$$

    But we know that this quantity is less than ϵ^2. If $|\lambda_i - \lambda| \geq \epsilon$ for each i, then we get that $a_1^2 + \ldots + a_n^2 < 1$, a contradiction. Therefore there is some i for which $|\lambda_i - \lambda| < \epsilon$, as required.

15. From the real spectral theorem, we know that if T is a self-adjoint linear operator, then there is a basis of U comprised of eigenvectors of T.

    If U has a basis $\{u_1, \ldots, u_n\}$ comprised of eigenvectors of T, then define an inner product on U by $\langle u_i, u_j \rangle = \delta_{ij}$. (This is the Kronecker delta function. Why can we define an inner product like this? Could you perform the calculations required to check the inner product axioms?)

    The matrix of T with respect to this basis is diagonal, as each u_i is an eigenvector of T. The basis $\{u_1, \ldots, u_n\}$ is orthonormal, by the definition of the inner product. Therefore having a diagonal (real) matrix implies that T is self-adjoint.

32. (a) The given formula for T, that

$$Tv = s_1\langle v, e_1\rangle f_1 + \ldots + s_n\langle v, e_n\rangle f_n,$$

is equivalent to saying that for each i, $Te_i = s_i f_i$. We know that each e_i is an eigenvector of T^*T, with eigenvalue s_i^2. Therefore for each i, $T^* s_i f_i = s_i^2 e_i$, and so $T^* f_i = s_i e_i$. Using a similar equivalence to that in the first line of this paragraph, this gives the required equation for T^*.

    (b) If any of the s_i are equal to zero, then the corresponding e_i is in the kernel of T, so T is not invertible. If none of the s_i are zero, then we may consider the map T^{-1} as defined in the question. We check that $T^{-1}Te_i = \frac{s_i}{s_i} e_i = e_i$ for each e_i and that $TT^{-1} f_i = \frac{s_i}{s_i} f_i = f_i$ for each f_i, so the map T^{-1} is the inverse of T, as required.

Other problems:

1. (a) By definition, we have that

$$\begin{aligned}\mathrm{ev}(aF + bG, v) &= (aF + bG)(v) \\ &= aF(v) + bG(v) \\ &= a\,\mathrm{ev}(F, v) + b\,\mathrm{ev}(G, v)\end{aligned}$$

The map F, as an element of $V^* = \mathcal{L}(V, \mathbb{F})$, is linear. Therefore

$$\begin{aligned}\mathrm{ev}(F, av + bw) &= (F)(av + bw) \\ &= aF(v) + bF(w) \\ &= a\,\mathrm{ev}(F, v) + b\,\mathrm{ev}(F, w)\end{aligned}$$

We have shown that ev is a bilinear map. By a result from the previous homework set, we have that ev induces a map from the tensor product $V^* \otimes V$ to $\mathbb{F}$. For each i and j, we have that $\mathrm{ev}(v_i^*, v_j) = v_i^*(v_j) = \delta_{ij}$.

   (b) For any $f \in V^*$ and any $w \in W$, let $\alpha(f \otimes w)$ be an element of $\mathcal{L}(V, W)$ defined by $\alpha(f \otimes w)(v) = f(v)w$. (Which property of tensor products guarantees us that this defines a linear map? This needs the definition of α to be bilinear on $V^* \times W$. Could you check this?).

   When V and W are finite dimensional, the map α will be an isomorphism. To check this, it suffices to show that the image of a basis of the domain is a basis of the codomain.

   The image of a pure tensor, $\alpha(v_i^* \otimes w_j)$, is the map that takes v_i to w_j and each other v_k to zero. But we have previously shown that these maps are a basis of $\mathcal{L}(V, W)$. Therefore the map α is an isomorphism.

   (c) For each i and j, let the (i, j)–entry of the matrix of T with respect to the basis $\{v_1, \ldots, v_n\}$ be a_{ij}. By examining the action of T on each basis vector v_i, we see that

$$\alpha^{-1}(T) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} v_j^* \otimes v_i.$$

Applying the map ev, we get that the trace of T is given by

$$\operatorname{tr}(T) = \sum_{i=1}^{n} a_{ii}.$$

This equals the trace of the matrix of T with respect to the given basis, as required.

(d) We will work with the matrices of these maps with respect to any fixed basis. Let the entries of the matrices of TS, T, S and ST be a_{ij}, b_{ij}, c_{ij} and d_{ij}, respectively. Then we calculate the following

$$\begin{aligned}
\operatorname{tr}(TS) &= \sum_{i=1}^{n} a_{ii} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}c_{ji} \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ji}b_{ij} \\
&= \sum_{j=1}^{n} d_{jj} \\
&= \operatorname{tr}(ST)
\end{aligned}$$

2. To check that $\langle \cdot, \cdot \rangle$ is an inner product, we calculate the following:

$$\begin{aligned}
\langle S, T \rangle &= \sum_{i=1}^{n} \langle Se_i, Te_i \rangle \\
&= \sum_{i=1}^{n} \overline{\langle Te_i, Se_i \rangle} \\
&= \overline{\langle T, S \rangle} \\
\langle aS + bS', T \rangle &= \sum_{i=1}^{n} \langle (aS + bS')e_i, Te_i \rangle \\
&= \sum_{i=1}^{n} \langle aSe_i + bS'e_i, Te_i \rangle \\
&= a\sum_{i=1}^{n} \langle Se_i, Te_i \rangle + b\sum_{i=1}^{n} \langle S'e_i, Te_i \rangle \\
&= a\langle S, T \rangle + b\langle S', T \rangle \\
\langle S, S \rangle &= \sum_{i=1}^{n} \langle Se_i, Se_i \rangle \\
&\geq 0, \text{ with equality only if each } Se_i = 0.
\end{aligned}$$

If each $Se_i = 0$, then $S = 0$, so $\langle S, S \rangle = 0$ only if $S = 0$. Therefore $\langle \cdot, \cdot \rangle$ is an inner product, as required.

Finally, we calculate that

$$\begin{aligned}
\langle S, T \rangle &= \sum_{i=1}^{n} \langle Se_i, Te_i \rangle \\
&= \sum_{i=1}^{n} \langle e_i, S^* T e_i \rangle \\
&= \operatorname{tr}(S^* T) \\
&= \operatorname{tr}(T S^*) \\
&= \langle T^*, S^* \rangle
\end{aligned}$$

Therefore the inner product $\langle S, T \rangle$ is independent of the choice of basis, as the trace of the operator $S^* T$ is independent of the choice of basis. Also, $\langle S, T \rangle = \langle T^*, S^* \rangle$.